Բարձր կարգի դիֆերենցիալներ

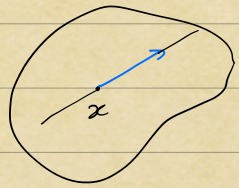
$X\subset\mathbb{R}^m$, $f: X\to\mathbb{R}$; $f\in C^k(X) \iff \forall l\in[\![1,k]\!]\ \forall\ i_1,\dots, i_l\in[\![1,m]\!]$

$$\exists\ \frac{\partial^l f}{\partial x_{i_l}\dots\partial x_{i_2}\partial x_{i_1}}\in C(X)$$

$\alpha_1=\#\{j\geq 1:\ i_j=1\}$, $\alpha_2=\#\{j\geq 1:\ i_j=2\},\dots,$ $\alpha_m=\#\{j\geq 1:\ i_j=m\}$; $\alpha_1+\dots+\alpha_m=l$

$$\frac{\partial^l f}{\partial x_1^{\alpha_1}\dots\partial x_m^{\alpha_m}}=:\partial^\alpha f,\qquad \alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{Z}_+^m,\quad |\alpha|=\alpha_1+\dots+\alpha_m$$

$$f\in C^k(X)\longrightarrow\qquad d^k f(x)[h]=\frac{d^k}{dt^k}f(x+th)\Big|_{t=0}\qquad f\text{-ի } k\text{-րդ դիֆերենցիալ}$$



Թեոր Դիցուք $f\in C^k(X)$ և $x\in X$: Այդ $\forall h\in\mathbb{R}^m$

$$d^k f(x)[h]=\sum_{\substack{|\alpha|=k\\ \alpha\in\mathbb{Z}_+^m}}\frac{k!}{\alpha!}\partial^\alpha f(x)h^\alpha,$$

որտեղ $\alpha!:=\alpha_1!\dots\alpha_m!$, $h^\alpha:=h_1^{\alpha_1}\dots h_m^{\alpha_m}$:

Ապացույց $\quad (b_1+\dots+b_m)^k=\sum\limits_{|\alpha|=k}\frac{k!}{\alpha!}b_1^{\alpha_1}\dots b_m^{\alpha_m}$

$$(b_1+\dots+b_m)^k=\underbrace{(b_1+\dots+b_m)\dots(b_1+\dots+b_m)}_{k \text{ անգամ}}=\sum_{\substack{\alpha\in\mathbb{Z}_+^m\\ |\alpha|=k}}C_{\alpha_1,\dots,\alpha_m}\,b_1^{\alpha_1}\dots b_m^{\alpha_m}$$

$\alpha_1,\dots,\alpha_m\in\mathbb{Z}_+$, $\alpha_1+\dots+\alpha_m=k$

$b_1^{\alpha_1}\dots b_m^{\alpha_m}$. $\quad b_1$-ի համար k փակագծից ընտրում α_1 հատ. $\binom{k}{\alpha_1}=\frac{k!}{\alpha_1!(k-\alpha_1)!}$

b_2-ի համար $k-\alpha_1$ փակագծից ընտրում ենք α_2 հատ. $\binom{k-\alpha_1}{\alpha_2}$

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b_j-ի համար. $\binom{k-\alpha_1\dots\alpha_{j-1}}{\alpha_j}=\frac{(k-\alpha_1\dots-\alpha_{j-1})!}{\alpha_j!(k-\alpha_1\dots-\alpha_j)!}$

$$C_{\alpha_1\dots\alpha_m}=\prod_{j=1}^m\binom{k-\alpha_1-\dots\alpha_{j-1}}{\alpha_j}=\frac{k!}{\alpha_1!\,\cancel{(k-\alpha_1)!}}\cdot\frac{\cancel{(k-\alpha_1)!}}{\alpha_2!\,\cancel{(k-\alpha_1-\alpha_2)!}}\cdots\frac{(k-\alpha_1-\dots\alpha_{m-1})!}{\alpha_m!\ 0!}$$

$$= \frac{k!}{\alpha_1! \cdots \alpha_m!} = \frac{k!}{\alpha!}$$

$$\frac{d}{dt} f(x+th)\Big|_{t=0} = \sum_{j=1}^{m} h_j \frac{\partial f}{\partial x_j} = (h_1 \partial_1 + \cdots + h_m \partial_m) f(x) \qquad \partial_j = \frac{\partial}{\partial x_j}$$

$$\frac{d^2}{dt^2} f(x+th)\Big|_{t=0} = \frac{d}{dt} \left[(h_1 \partial_1 + \cdots + h_m \partial_m) f \right](x+th)\Big|_{t=0}$$

$$= (h_1 \partial_1 + \cdots + h_m \partial_m)^2 f(x)$$

$$h_1 \frac{\partial f}{\partial x_1}(x) + h_2 \frac{\partial f}{\partial x_2}(x) + \cdots + h_m \frac{\partial f}{\partial x_m}(x) = (h_1 \partial_1 + \cdots + h_m \partial_m) f(x)$$

$$d^k f(x)[h] = (h_1 \partial_1 + \cdots + h_m \partial_m)^k f$$

$$= \underbrace{(h_1 \partial_1 + \cdots + h_m \partial_m) \times \cdots \times (h_1 \partial_1 + \cdots + h_m \partial_m)}_{k \text{ անգամ}} f$$

$$= \sum_{|\alpha| = k} \frac{k!}{\alpha_1! \cdots \alpha_m!} (h_1 \partial_1)^{\alpha_1} (h_2 \partial_2)^{\alpha_2} \cdots (h_m \partial_m)^{\alpha_m} f$$

$$= \sum_{|\alpha| = k} \frac{k!}{\alpha!} \underbrace{h_1^{\alpha_1} \cdots h_m^{\alpha_m}}_{h^\alpha} \underbrace{(\partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m} f)}_{\partial^\alpha f}$$

Օրինակներ $\quad \underline{k=1} \qquad \alpha = (0, \dots, \underset{\substack{\uparrow \\ j}}{1}, \dots 0)$

$$d f(x)[h] = \sum_{j=1}^{m} h_j \partial_j f(x)$$

$\underline{k=2} \qquad \alpha = (0, \dots, 2, \dots 0)$ կամ $\alpha = (0 \dots \overset{i}{1} \dots \overset{j}{1}\ 0 \dots)$

$$d^2 f(x)[h] = \sum_{j=1}^{m} h_j^2 \partial_j^2 f(x) + 2 \underbrace{\sum_{i<j}}_{\sum_{i \neq j}} h_i h_j \partial_i \partial_j f(x) = \sum_{i,j=1}^{m} \partial_i \partial_j f(x) h_i h_j$$

$$d^3 f(x)[h] = \sum_{j=1}^{n} h_j^3 \partial_j^3 f + 3 \sum_{i \neq j} h_i h_j^2 \partial_i \partial_j^2 f + 6 \sum_{i<j<e} h_i h_j h_e (\partial_i \partial_j \partial_e f)$$

$\alpha = (0 \dots 3 \dots 0), \quad \alpha = (\ 1 \ \ 2\ ), \quad \alpha = (\ 1 \ \ 1 \ \ 1\ )$

§12. Թեյլորի բանաձևը և լոկալ էքստրեմումներ

$m=1$

$f\in C^k(I)$

$$f(x+h)=\sum_{j=0}^{k}\frac{f^{(j)}(x)}{j!}h^j+o(h^k)$$

$$r(h)=f(x+h)-\sum_{j=0}^{k}\frac{f^{(j)}(x)}{j!}h^j,\qquad \lim_{h\to0}\frac{r(h)}{h^k}=0$$

Թեորեմ Դիցուք $f\in C^k(X)$, $k\geq 1$: Այդժամ $\forall x\in X$ $\forall h\in\mathbb{R}^m$ $x+h\in X$

$$f(x+h)=\sum_{j=0}^{k}\frac{d^jf(x)[h]}{j!}+r(h)\qquad \left(d^0f(x)=f(x)\right)$$

$$=\sum_{\substack{\alpha\in\mathbb{Z}_+^m\\|\alpha|\leq k}}\frac{1}{\alpha!}\partial^\alpha f(x)h^\alpha+r(h)$$

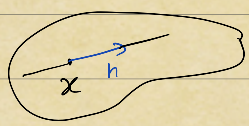
որտեղ $r(h)=o(|h|^k)$. $\lim\limits_{|h|\to0}\dfrac{|r(h)|}{|h|^k}=0$:

Ապացույց

$$\sum_{j=0}^{k}\frac{d^jf(x)[h]}{j!}=\sum_{j=0}^{k}\frac{1}{\cancel{j!}}\sum_{\substack{\alpha\in\mathbb{Z}_+^m\\|\alpha|=j}}\frac{\cancel{j!}}{\alpha!}\partial^\alpha f(x)h^\alpha$$

$$=\sum_{\substack{\alpha\in\mathbb{Z}_+^m\\|\alpha|\leq k}}\frac{1}{\alpha!}\partial^\alpha f(x)h^\alpha$$

$$g\in C^k([0,1]),\qquad g(x)-g(0)=\sum_{j=1}^{k-1}\frac{g^{(j)}(0)}{j!}x^j+\frac{1}{(k-1)!}\underbrace{\int_0^x g^{(k)}(t)(x-t)^{k-1}dt}$$



$$=\int_0^x g^{(k)}(t)d\left(-\frac{(x-t)^k}{k!}\right)$$

$$g(t)=f(x+th)$$

$$=-g^{(k)}(t)\frac{(x-t)^k}{k!}\Bigg|_{t=0}^{t=x}+\cdots=\frac{g^{(k)}(0)}{k!}x^k+\cdots$$

$$g(1)-g(0)=\sum_{j=1}^{k-1}\frac{g^{(j)}(0)}{j!}+\frac{1}{(k-1)!}\int_0^1 g^{(k)}(t)(1-t)^{k-1}dt$$

$$f(x+h)-f(x)=\sum_{j=0}^{k-1}\frac{d^jf(x)(h)}{j!}+\frac{1}{(k-1)!}\int_0^1 \underbrace{g^{(k)}(t)}_{\underbrace{g^{(k)}(0)}_{d^kf(x)(h)}+(g^{(k)}(t)-g^{(k)}(0))}(1-t)^{k-1}dt$$

$$=\sum_{j=0}^{k}\frac{d^jf(x)(h)}{j!}+r(h),$$

որտեղ

$$r(h) = \frac{1}{(k-1)!} \int_0^1 \underbrace{\left(g^{(k)}(t) - g^{(k)}(0)\right)}_{\sup\limits_{t\in(0,1)} = o(|h|^k)} (1-t)^{k-1}\, dt$$

Բավական է ապացուցել, որ $\sup\limits_{t \in (0,1)} |g^{(k)}(t) - g^{(k)}(0)| = o(|h|^k)$

$$g^{(k)}(t) = \frac{d^k}{dt^k} f(x+th) = (h_1 \partial_1 + \dots + h_m \partial_m)^k f(x+th)$$

$$|g^{(k)}(t) - g^{(k)}(0)| = |(h_1 \partial_1 + \dots + h_m \partial_m)^k (f(x+th) - f(x))|$$

$$= \Big| \sum_{|\alpha|=k}' \frac{1}{\alpha!} h^\alpha (\partial^\alpha f(x+th) - \partial^\alpha f(x)) \Big|$$

$h^\alpha = h_1^{\alpha_1} \cdots h_m^{\alpha_m}$

$$\le C_k |h|^k \max_{|\alpha|=k} |\partial^\alpha f(x+th) - \partial^\alpha f(x)|$$

$$\sup_{|h|\le\delta} \frac{\sup\limits_{t\in(0,1)} |g^{(k)}(t) - g^{(k)}(0)|}{|h|^k} \le C_k \sup_{\substack{|\alpha|=k \\ |h|\le\delta}} |\partial^\alpha f(x+h) - \partial^\alpha f(x)| \xrightarrow[\delta\to 0]{} 0$$

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Լեմմ Թեյլորի բազմանդամի միակությունը. $f \in C^k(X)$, $x \in X$

$$f(x+h) - f(x) = \sum_{0<|\alpha|\le k}' c_\alpha h^\alpha + o(|h|^k) \Rightarrow \forall \alpha \quad c_\alpha = \frac{\partial^\alpha f(x)}{\alpha!}$$

Ապացույց

$$\sum_{0<|\alpha|\le k} \underbrace{\left(c_\alpha - \frac{\partial^\alpha f(x)}{\alpha!}\right)}_{b_\alpha} h^\alpha + o(|h|^k) = 0 \Rightarrow \forall \alpha \quad b_\alpha = 0$$

$$\sum_{|\alpha|\le k}' b_\alpha h^\alpha = o(|h|^k) \Rightarrow \forall \alpha \quad b_\alpha = 0$$

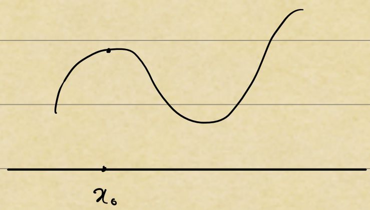
$$b_0 = o(h) \Rightarrow b_0 = 0$$

$$\sum_{|\alpha|=1} b_\alpha h^\alpha = o(|h|) \Rightarrow b_1 h_1 + \dots + b_m h_m = o(|h|) \Rightarrow b_1 = \dots = b_m = 0$$

$b_\alpha = 0, \quad |\alpha| < j \quad \Rightarrow \quad \sum_{|\alpha| = j} b_\alpha h^\alpha = o(|h|^j)$

$$\overset{h = th'}{\Rightarrow} \quad \sum_{|\alpha|=j} b_\alpha t^j (h')^\alpha = o(t^j) \Rightarrow \sum_{|\alpha|=j} b_\alpha h^\alpha = 0 \quad \blacksquare$$

Մի քանի փոփոխականի ֆունկցիաների էքստրեմումներ



$f \in C^2(I), \quad x_0 \in I$

- $f'(x_0) = 0, \quad f''(x_0) \leq 0$ (մաքս)
- $f'(x_0) = 0, \quad f''(x_0) < 0 \Rightarrow x_0$-ն լոկալ մաքսիմում f

$$f(x_0 + h) - f(x_0) = \frac{1}{2} f''(x_0) h^2 + o(h^2)$$

Քառակուսային ձևեր. $\quad f(x_0 + h) - f(x_0) = \frac{1}{2} \sum_{i,j=1}^{m} \partial_i \partial_j f(x_0) h_i h_j + o(|h|^2)$

$$\sum_{i,j=1}^{m} a_{ij} h_i h_j, \qquad a_{ij} = a_{ji}$$