Last update: 10:11pm (Yerevan), Apr 7.

Mathematical Logic HOMEWORK 7 **Due: Apr 17 (Wed)**

1.[*] Prove the following theorems, noting first that Lovász's theorem is a special case of Łoś–Tarski. Nonetheless, I suggest proving Lovász's theorem first because it's more approachable.

(a) **Łoś–Tarski Theorem.** Let σ be a finite signature and $\mathcal{C}$ be a finitely axiomatizable class of σ-structured. Then $\mathcal{C}$ is closed under finitely generated substructures[1] if and only if $\mathcal{C}$ is axiomatized by a universal σ-sentence.

(b) Let $k \in \mathbb{N}$ and call a (potentially infinite) graph k-**coverable** if it admits $\leqslant k$ vertices such that each edge is incident to at least one of them.

Lovász's Theorem. For each $k \in \mathbb{N}$, there exists finitely many finite graphs $H_1, H_2, \dots, H_m$ (forbidden patterns) such that for every graph G, we have that G is k-coverable if and only if it does not contain any of the H_i as a subgraph.

HINT: First prove that if every finite subcover of a graph G is k-coverable, then such is G. Then take the collection of minimal counter-examples to k-coverability and prove that this collection has to be finite.

REMARK: Lovász's original proof is

2. Let $\sigma_S := (0, S)$ where 0 is a constant symbol and S is a unary function symbol. Let T_S be the σ_S-theory consisting of the following (infinitely-many) axioms:

(S1) Zero has no predecessor: $\forall x(S(x) \neq 0)$.

(S2) The successor function is one-to-one: $\forall x \forall y(S(x) = S(y) \to x = y)$.

(S3) Any nonzero number is a successor of something: $\forall x(x \neq 0 \to \exists y(x = S(y)))$.

($S4^\infty$) There are no cycles: $\varphi_n := \forall x(S^n(x) \neq x)$ for every $n \in \mathbb{N}$.

(a) Observe that $\boldsymbol{N}_S := (\mathbb{N}, 0, S)$ is a (standard) model of T_S and describe all models of T_S.

(b) Prove that T_S is κ-categorical for each uncountable cardinal κ, and deduce that T_S is complete.

3. Let K be a field and let $\overline{K}$ be an algebraic closure of K. A nonconstant polynomial $f \in K[X_1, ..., X_n]$ is called *irreducible* if whenever $f = gh$ for some $g, h \in K[X_1, ..., X_n]$, either $\deg(g) = 0$ or $\deg(h) = 0$. Furthermore, f is called *absolutely irreducible* if it is irreducible in $\overline{K}[X_1, ..., X_n]$ (view f as an element of $\overline{K}[X_1, ..., X_n]$).

For example, the polynomial $X^2 + 1 \in \mathbb{R}[X]$ is irreducible, but it is not absolutely irreducible since $X^2 + 1 = (X + i)(X - i)$ in $\mathbb{C}[X]$. On the other hand, $XY - 1 \in \mathbb{Q}[X, Y]$ is absolutely irreducible.

[1] By this we mean that if a σ-structure is in $\mathcal{C}$ then so are all of its finitely generated substructures.

Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and prove the following:

Theorem (Noether–Ostrowski Irreducibility Theorem)**.** *For $f \in \mathbb{Z}[X_1, \ldots, X_n]$ and prime p, let f_p denote the polynomial in $\mathbb{F}_p[X_1, \ldots, X_n]$ obtained by applying the canonical map $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ to the coefficients of f (i.e. mod-ing out the coefficients by p). For all $f \in \mathbb{Z}[X_1, \ldots, X_n]$, f is absolutely irreducible (as an element of $\mathbb{Q}[X_1, \ldots, X_n]$) if and only if f_p is absolutely irreducible (as an element of $\mathbb{F}_p[X_1, \ldots, X_n]$) for all sufficiently large primes p.*

Hint: Your proof should be shorter than the statement of the theorem.
Remark: The original algebraic proof of this theorem is quite involved.