Math Logic: Model Theory & Computability
Lecture 21

Equivalently, we have the following syntactic version of Compactness:

Syntactic Compactness. If every finite subtheory of a σ-theory T is consistent, then T is consistent.

Proof. If T is inconsistent then $T \vdash \perp$ and since proofs are finite, some finite subtheory $T_0 \subseteq T$ proves $\perp$, and therefore T_0 is inconsistent. $\square$

Cor (from syntactic compactness). Let I be a set (of indices) and let $\{T_i\}_{i\in I}$ be a nested collection of σ-theories, i.e. for any $i,j \in I$, $T_i \subseteq T_j$ or $T_j \subseteq T_i$. If each T_i is consistent, then so is $\bigcup_{i\in I} T_i$.

Proof. HW. $\square$

Enriching consistent theories.

Lemma about consistency and extensions. Let T be a σ-theory.

(a) For a σ-sentence φ, the theory $T \cup \{\varphi\}$ is inconsistent iff $T \vdash \neg\varphi$.

(b) If T is consistent, then for any σ-sentence φ, $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is consistent (possibly both).

(c) Adding a Henkin witness: For an extended σ-formula $\varphi(v)$, if $T \cup \{\exists v \varphi\}$ is consistent and c is a constant symbol in σ that doesn't appear in $T \cup \{\exists v \varphi\}$, then $T \cup \{\varphi(c/v)\}$ is consistent. (Thus, $T \cup \{\exists v \varphi, \varphi(c/v)\}$ is also consistent.)

Proof. (a) $\Leftarrow$. If $T \vdash \neg\varphi$ then $T \cup \{\varphi\} \vdash \neg\varphi$ and $T \cup \{\varphi\} \vdash \varphi$, so $T \cup \{\varphi\}$ is inconsistent.

$\Rightarrow$. Suppose $T \cup \{\varphi\}$ is inconsistent, so $T \cup \{\varphi\} \vdash \neg\top$. Then $T \vdash \varphi \to \neg\top$, so because $\vdash (\varphi \to \neg\top) \to (\top \to \neg\varphi)$, we get $T \vdash \top \to \neg\varphi$. Because also $T \vdash \top$, we get by MP that $T \vdash \neg\varphi$.

(b) Towards the contrapositive, suppose both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are inconsistent. Then by (a), $T \vdash \neg\varphi$ and $T \vdash \neg\neg\varphi$, so T is inconsistent.

(c) We prove the contrapositive. Suppose $T \cup \{\varphi(c/v)\}$ is inconsistent, so by (a), $T \vdash \neg\varphi(c/v)$. By Constant Substitution Lemma, $T \vdash \neg\varphi$, so by generalization axiom (5), $T \vdash \forall v \neg\varphi$, i.e. $T \vdash \neg\exists v\varphi$ hence $T \cup \{\exists v\varphi\}$ is inconsistent.

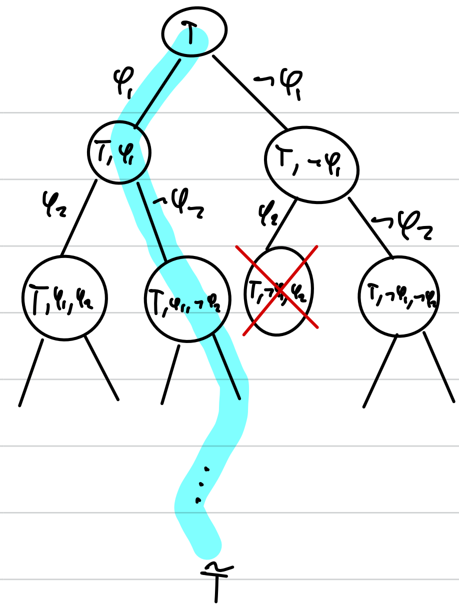
From this it follows that $T \cup \{\varphi(c/v), \exists v\varphi\}$ is consistent and this left as HW. $\square$

Prop. Every consistent σ-theory T admits a σ-maximal consistent extension $\tilde{T} \supseteq T$.

Proof 1. It follows from the corollary of syntactic compactness that if $(T_i)_{i\in I}$ is an increasing chain of consistent theories, then $\bigcup_{i\in I} T_i$ is a consistent theory, so Zorn's lemma applies and gives an inclusion maximal consistent σ-theory $\tilde{T} \supseteq T$. This $\tilde{T}$ is σ-maximal consistent because for each σ-sentence φ, one of $\tilde{T} \cup \{\varphi\}$ and $\tilde{T} \cup \{\neg\varphi\}$ is consistent by part (b) of the above lemma, so by inclusion maximality of $\tilde{T}$, $\varphi \in \tilde{T}$ or $\neg\varphi \in \tilde{T}$. $\square$

Proof 2. We only prove for a ctbl signature σ to avoid transfinite recursion. Suppose σ is ctbl, hence Sentences(σ) is ctbl and we fix an enumeration $(\varphi_n)_{n\geq 1}$ of all σ-sentences. We inductively define an increasing sequence $(T_n)_{n\geq 0}$ of consistent σ-theories with $T_0 := T$, so that for each $n \geq 1$, either $\varphi_n \in T_n$ or $\neg\varphi_n \in T_n$. Let $T_0 := T$ and suppose T_n is defined for $n \geq 0$, and we define T_{n+1} as $T_{n+1} := T_n \cup \{\varphi_n\}$ if $T_n \nvdash \neg\varphi_n$, and $T_{n+1} := T_n \cup \{\neg\varphi_n\}$ if $T_n \vdash \neg\varphi_n$.

Then by the corollary of syntactic compactness, $\tilde{T} := \bigcup_{n \in \mathbb{N}} T_n$ is consistent. Moreover, for each $n \geq 1$, $\varphi_n \in \tilde{T}$ or $\neg\varphi_n \in \tilde{T}$, so $\tilde{T}$ is σ-maximal consistent.



Syntactic - Semantic Duality Theorem.

We aim to prove the following theorem, which is called the Completeness of First-Order Logic or just Gödel's Completeness Theorem (not to be confused with the completeness of a particular first-order theory).

Gödel's Completeness Theorem. Every consistent σ-theory is satisfiable.

This is equivalent to the following theorem:

Syntactic - Semantic Duality. For every σ-theory T and σ-sentence φ,
$$T \models \varphi \quad \text{iff} \quad T \vdash \varphi.$$

Gödel Completeness $\Rightarrow$ S-S Duality. We already showed that $T \vdash \varphi$ implies $T \models \varphi$. To prove the converse, we prove the contrapositive: suppose $T \nvdash \varphi$, then by (a) of the above lemma about consistency, $T \cup \{\neg\varphi\}$ is consistent, hence by Gödel Completeness has a model $\underline{M} \models T \cup \{\neg\varphi\}$, so $T \not\models \varphi$ here $\underline{M} \models T$ and $\underline{M} \not\models \varphi$. $\square$

S-S Duality $\Rightarrow$ Gödel Completeness. By duality, $T \models \bot$ iff $T \vdash \bot$, which just says that T is satisfiable iff T is consistent. $\square$

Remark. We have proved Ax's theory for $\mathbb{C}$, which implies, by the completeness of ACF_0 that $ACF_0 \models$ Ax's theorem for fixed degree. This proof uses non-first-order arguments like the compactness theorem, set theory, pigeonhole principle, etc, but the duality theorem says that $ACF_0 \vdash$ (Ax's theorem for fixed degree), so our fancy proof was an overkill. However, we haven't found this first-order proof explicitly.

HW Build a model for a sentence in a finite signature that asserts the existence of 5 elements and describes how each constant symbol, relation symbol, and function symbol is defined.

Caution. $\exists x_1 \exists x_2 \ldots \exists x_5 (\ldots x_1 = x_2 \ldots)$ implies that a model of this should have at most 4 elements.