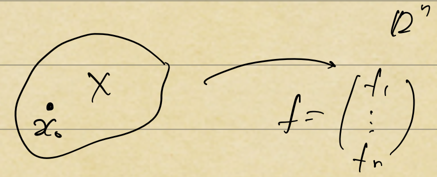
§11. Դիֆերենցելի արտապատկերումներ

$X \subset \mathbb{R}^m, \quad f: X \to \mathbb{R}^n, \quad x_0 \in X$

Սահմանում f-ը դիֆերենցելի է x_0 կետում $\Longleftrightarrow \exists A \in L(\mathbb{R}^m, \mathbb{R}^n)\ \forall h \in \mathbb{R}^m$
$x_0 + h \in X \quad f(x_0+h) - f(x_0) = Ah + r(h),$
որտեղ $r(h) = o(|h|)$

$$\lim_{h \to 0} \frac{|r(h)|}{|h|} = 0$$

Նշանակում $A = Df(x_0) = df(x_0) = f'(x_0)$



- $f = (f_1, \dots f_n)$, f-ը դիֆերենցելի $\Longleftrightarrow$ $f_1, \dots, f_n$ դիֆերենցելի են

- f-ը դիֆերենցելի է $\Rightarrow \forall k \in [1, m]$ $\exists \dfrac{\partial f}{\partial x_k}(x_0) = \lim\limits_{t \to 0} \dfrac{f(x_0 + te_k) - f(x_0)}{t}$

$\exists \dfrac{\partial f}{\partial x_k} \in C(X)\ \forall k \in [1, m] \Rightarrow$ f-ը դիֆերենցելի է X-ում.

$f = (f_1, \dots f_n)$, $\quad f_1'(x_0) = Df(x_0) \in L(\mathbb{R}^m, \mathbb{R})$

$\dfrac{\partial f_1}{\partial x_k}(x_0)$ մասնակի ածանցյալ $\in \mathbb{R}$

$$\frac{\partial f}{\partial x_1} = \left(\frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_n}{\partial x_1} \right)$$

Թեորեմ ա) $f, g: X \subset \mathbb{R}^m \to \mathbb{R}^n$ ածանցելի են $x \in X$ կետում, $c \in \mathbb{R}$:
Ապա $f + cg$ ածանցելի է x-ում և $(f+cg)'(x) = f'(x) + cg'(x)$

բ) $n = 1 \Rightarrow (fg)'(x) = g(x) f'(x) + f(x) g'(x)$

գ) $n = 1, \quad g(x) \neq 0 \Rightarrow \left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}$

Ապացույց ա) $(f+cg)(x+h) - (f+cg)(x) = (f'(x) + cg'(x))h + o(|h|)$

$$\left(\cancel{f(x)} + f'(x)h + o(|h|)\right) + c\left(\cancel{g(x)} + g'(x)h + o(|h|)\right) - \left(\cancel{f(x)} + c\cancel{g(x)}\right) = f'(x)h + cg'(x)h + o(|h|)$$

g) $x \to \dfrac{1}{g(x)}$

$$\frac{1}{g(x+h)} - \frac{1}{g(x)} = \frac{1}{g(x)+g'(x)h+o(|h|)} - \frac{1}{g(x)}$$

$$= \frac{g(x)-(g(x)+g'(x)h+o(|h|))}{g(x)(g(x)+g'(x)h+o(|h|))}$$

$$= \underbrace{-\frac{g'(x)h+o(|h|)}{g(x)^2+g(x)g'(x)h+o(|h|)}}_{\mathrm{I}} \overset{?}{=} \underbrace{-\frac{g'(x)h}{g(x)^2}}_{\mathrm{II}} + o(|h|)$$

$$\mathrm{I}-\mathrm{II} = \frac{-\cancel{g(x)^2 g'(x)h} + \cancel{g(x)^2 g'(x)h} + g(x)(g'(x)h)^2 + o(|h|)}{g(x)^2(g(x)^2+g(x)g'(x)h+o(|h|))}$$

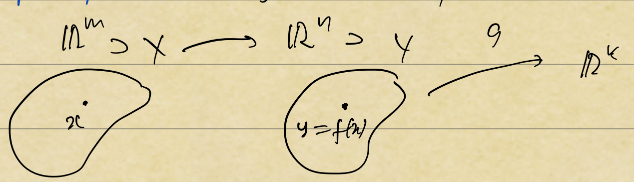
$$\frac{|g(x)(g'(x)h)^2|}{|h|} \le \frac{C|h|^2}{|h|} = C|h| \to 0, \quad |h| \to 0 \quad \blacksquare$$

- $A \in L(\mathbb{R}^m, \mathbb{R}^n)$, $B \in L(\mathbb{R}^n, \mathbb{R}^k)$ $\longrightarrow$ $B \circ A \in L(\mathbb{R}^m, \mathbb{R}^k)$

  $B \circ A = BA$

- $f'(x) = \left(\dfrac{\partial f_i}{\partial x_j}(x)\right)_{\substack{1\le i\le n\\ 1\le j\le m}}$, $\quad (f'(x)h)_i = \displaystyle\sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j}(x)\, h_j$

Теорема $f: X \to \mathbb{R}^n$, $g: Y \to \mathbb{R}^k$, $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$; $x \in X$, $f(x) = y \in Y$



Да предположим, че f-я диференцируема в x-а и g-я диференцируема в y-а. Тогава $\exists\, U \ni x$ $f(U) \subset Y$ и $g \circ f: U \to \mathbb{R}^k$ диференцируема в x-а, като при това

$$\underset{\mathbb{R}^m \to \mathbb{R}^k}{(g \circ f)'(x)} = \underset{\mathbb{R}^n \to \mathbb{R}^k}{g'(f(x))} \circ \underset{\mathbb{R}^m \to \mathbb{R}^n}{f'(x)}$$

Доказателство

$$f(x+h) = f(x) + f'(x)h + r_1(h), \quad r_1(h) = o(|h|)$$

$$g(y+k) = g(y) + g'(y)k + r_2(k), \quad r_2(k) = o(|k|)$$

$$\underbrace{g(f(x+h))}_{(g\circ f)(x+h)} = g(\underbrace{f(x)}_{=y} + \underbrace{f'(x)h + r_1(h)}_{=k})$$

$$= g(f(x)) + g'(f(x))\left(f'(x)h + r_1(h)\right) + r_2\left(f'(x)h + r_1(h)\right)$$

$$= \underbrace{g(f(x))}_{(g\circ f)(x)} + \underbrace{g'(f(x))\,f'(x)h}_{g'(y)\circ f'(x)} + r(h),$$

որտեղ $\quad r(h) = g'(f(x))r_1(h) + r_2\left(f'(x)h + r_1(h)\right) \overset{?}{=} o(|h|)$

$$\frac{|g'(f(x))r_1(h)|}{|h|} \le \frac{\|g'(y)\|\,|r_1(h)|}{|h|} \to 0, \quad \text{երբ } |h| \to 0:$$

$$\|A\| = \sup_{|h|\le 1} |Ah|$$

$$\frac{|r_2(Ah + r_1(h))|}{|h|} = \underbrace{\frac{|r_2(Ah + r_1(h))|}{|Ah + r_1(h)|}}_{\to 0,\ |h|\to 0} \times \underbrace{\frac{\overbrace{|Ah + r_1(h)|}^{\le \|A\|\cdot|h| + o(|h|)}}{|h|}}_{\le C} \to 0, \quad |h| \to 0 \quad \blacksquare$$

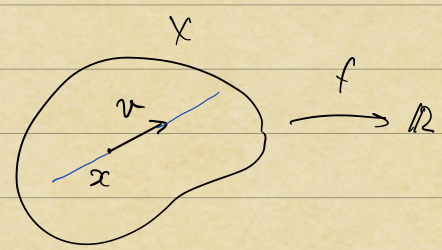
Հետևանք $\quad (g\circ f)'(x) = \left(\frac{\partial (g\circ f)_i}{\partial x_j}\right)_{\substack{1\le i\le k\\ 1\le j\le m}} = g'(y)\cdot f'(x) = \sum_{e=1}^{n} \frac{\partial g_i}{\partial y_e}(f(x))\frac{\partial f_e}{\partial x_j}(x)$

$$\frac{\partial (g\circ f)_i}{\partial x_j}(x) = \sum_{e=1}^{n} \frac{\partial g_i}{\partial y_e}(f(x))\,\frac{\partial f_e}{\partial x_j}(x)$$

Օրինակ $\quad f: \mathbb{R}^n \to \mathbb{R}, \quad a: I \to \mathbb{R}^n, \quad g(t) = f(a(t)) = f(a_1(t), \dots, a_n(t)), \quad g: I \to \mathbb{R}$

$$g'(t) = \sum_{e=1}^{n} \frac{\partial f}{\partial x_e}(a(t))\, a_e'(t)$$

Ածանցյալ ըստ ուղղության

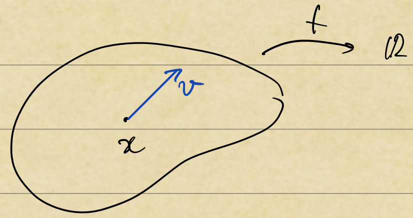


$$X \subset \mathbb{R}^m, \qquad f: X \to \mathbb{R}, \qquad v \in \mathbb{R}^m$$

$$D_v f(x) = \frac{d}{dt} f(x + tv)\Big|_{t=0}$$

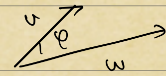
$$= \frac{d}{dt} f(x_1 + tv_1, \dots, x_m + tv_m)\Big|_{t=0}$$

$$= \sum_{j=1}^{m} \frac{\partial f}{\partial x_j}(x)\, v_j = (\nabla f(x), v), \quad \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_m}(x)\right) \text{ գրադիենտ}$$



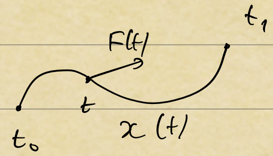
$$D_v f(x) = (\nabla f(x), v), \quad |v| = 1$$

$$(w, v) = |w| \cdot |v| \cos\varphi$$

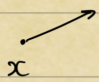


Հետևություն. Ֆունկցիայի աճը առավելագույնն է $\dfrac{\nabla f(x)}{|\nabla f(x)|}$ վեկտորի ուղղությամբ

Կիրառություն



$$\frac{m|\dot x(t_1)|^2}{2} - \frac{m|\dot x(t_0)|^2}{2} = W = \int_{t_0}^{t_1} (F(t), \dot x(t))\,dt$$



$$F(x) = (F_1(x), \dots, F_m(x))$$

$$\frac{1}{2} m|\dot x(t_1)|^2 - \frac{1}{2} m|\dot x(t_0)|^2 = \int_{t_0}^{t_1} (F(x(t)), \dot x(t))\,dt$$

Սահմանում Կասենք, որ $F(x)$ դաշտը <u>պոտենցիալ</u> է, եթե $\exists\, V \in C^1(\mathbb{R}^m)$

$$\forall x \in \mathbb{R}^m \qquad F(x) = -\nabla V(x) \iff F_j(x) = -\frac{\partial V}{\partial x_j}(x)$$

$$\Downarrow$$

$$\frac{\partial F_i}{\partial x_i}(x) = -\frac{\partial}{\partial x_i}\left(\frac{\partial V}{\partial x_j}(x)\right)$$

$$(V \in C^2) \quad = -\frac{\partial}{\partial x_j}\left(\frac{\partial V}{\partial x_i}(x)\right)$$

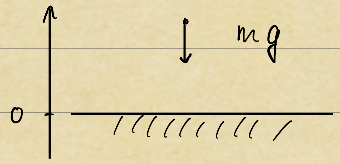
$$\frac{1}{2} m|\dot x(t_1)|^2 - \frac{1}{2} m|\dot x(t_0)|^2 = \int_{t_0}^{t_1} \underbrace{(-\nabla V(x(t)), \dot x(t))}_{-\sum_{j=1}^{m} \frac{\partial V}{\partial x_j}(x(t))\,\dot x_j(t) = -\frac{d}{dt} V(x(t))}\,dt$$

$$= -\int_{t_0}^{t_1} \frac{d}{dt} V(x(t))\,dt = -V(x(t_1)) + V(x(t_0))$$

$$\forall t_0 < t_1 \qquad \frac{1}{2} m|\dot x(t_1)|^2 + V(x(t_1)) = \frac{1}{2} m|\dot x(t_0)|^2 + V(x(t_0))$$

$$E(x, \dot x) = \frac{1}{2} m|\dot x|^2 + V(x)$$ էներգիան չի փոխվում հետագծերի երկայնքով

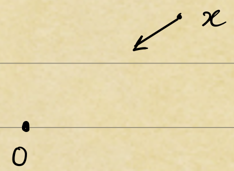
[illegible]

1) 

$$F(x) = -mg$$

$$V'(x) = -F(x) = mg \Rightarrow V(x) = mgx$$

$$\frac{1}{2} m \dot{x}^2 + mgx$$ [illegible]

2) 

$$F(x) = -\frac{C}{|x|^3} x, \quad |F(x)| = \frac{C}{|x|^2}$$

$$V(x) = \frac{C}{|x|}, \quad \nabla V(x) = -F(x), \quad x \neq 0$$

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2}, \quad V(x) = \frac{C}{(x_1^2 + \cdots + x_n^2)^{1/2}} = C(|x|^2)^{-\frac{1}{2}}$$

$$\frac{\partial V}{\partial x_j} = C \times \left(-\frac{1}{2}\right) |x|^{-3} \times 2x_j = -\frac{C x_j}{|x|^3} = F_j(x)$$

$$\frac{\partial V}{\partial x_j} = C \times \left(-\frac{1}{2}\right) \underbrace{(|x|^2)^{-\frac{3}{2}}}_{|x|^{-3}} \underbrace{\left(\frac{\partial}{\partial x_j} |x|^2\right)}_{2x_j}$$

$$V(x) = |x|^{-1}, \quad \frac{\partial V}{\partial x_j} = -C|x|^{-2} \frac{\partial}{\partial x_j} |x| = -|x|^{-2} \frac{2x_j}{2|x|} \times C$$