Mathematical Logic HOMEWORK 6 **Due: Apr 3 (Wed)**

1. Show that the following classes of structures are not axiomatizable, namely:

(a) Cycle graphs, i.e. undirected graphs that look like an undirected cycle of some length.

(b) Non-bipartite graphs.

(c) Groups that contain elements of arbitrarily large finite order.

(d) Torsion group, i.e. groups in which every element has a finite order.

2. Overspill. Let $\boldsymbol{M}$ be a nonstandard model of PA, let $\varphi(x,\vec{y})$ be an extended σ_{arthm}-formula, where $|\vec{y}| = k$, and let $\vec{a} \in M^k$. Show that if $\boldsymbol{M} \models \varphi(n,\vec{a})$ for infinitely many $n \in \mathbb{N}^{\boldsymbol{M}}$, then there is $w \in M \setminus \mathbb{N}^{\boldsymbol{M}}$ such that $\boldsymbol{M} \models \varphi(w,\vec{a})$. In other words, no infinite subset of $\mathbb{N}^{\boldsymbol{M}}$ is definable in $\boldsymbol{M}$; in particular, $\mathbb{N}^{\boldsymbol{M}}$ itself is not definable.

3. Let $\boldsymbol{M}$ be a nonstandard model of PA.

(a) For all $a, b \in M$, define

$$a \sim b \iff |a-b| \in \mathbb{N}^{\boldsymbol{M}},$$

where $z = |a-b|$ is the unique element in M such that $a+z=b$ or $b+z=a$. Show that $\sim$ is an equivalence relation on M and that it is NOT definable in $\boldsymbol{M}$.

(b) Let $Q := M/\sim$ denote the quotient by this equivalence relation, i.e. $Q := \{[a] : a \in M\}$, where $[a]$ denotes the equivalence class of a. Define the relation $<_Q$ on Q as follows: for all $[a],[b] \in Q$,

$$[a] <_Q [b] \iff \text{there is } c \in M \setminus \mathbb{N}^{\boldsymbol{M}} \text{ such that } a+c=b.$$

Show that $<_Q$ is well-defined (does not depend on the representatives a, b) and is a strict linear order on Q.

(c) Show that the order $(Q, <_Q)$ has a least element but no greatest element, and it is a dense (in itself), i.e. $u <_Q v \implies \exists w(u <_Q w <_Q v)$ for all $u, v \in Q$. Thus, $(Q, <_Q)$ is isomorphic to $(\mathbb{Q}_{\geqslant 0}, <)$.

4. Let $\sigma_{\text{gph}} := (E)$ be the signature for graphs and let

$$T := \left\{\varphi_{\text{smpl}}, \varphi_{\text{2reg}}\right\} \cup \{\varphi_n : n \in \mathbb{N}^+\},$$

where φ_{smpl} says that E is symmetric and irreflexive (i.e. the graph is simple), φ_{2reg} says that every vertex has exactly 2 neighbours (i.e. the graph is 2-regular), and φ_n says that there is no cycle of length n.

(a) Observe that every model of T is a graph whose connected components are bi-infinite lines (let's call them $\mathbb{Z}$-lines).

(b) Prove that two models of T are isomorphic if and only if they have equinumerous sets of connected components (i.e. the sets of connected components have equal cardinality).

(c) Conclude that any two uncountable models of T of the same cardinality are isomorphic.

HINT: This uses our usual blackbox from cardinal arithmetic: $|A \times B| = \max(|A|, |B|)$ for sets A, B, at least one of which is infinite.

(d) Prove that T is complete.

HINT: Recall that T is complete if and only if any two models $\boldsymbol{A}, \boldsymbol{B}$ of T have the same theory. Use some Löwenheim–Skolem theorem to upgrade the given models $\boldsymbol{A}, \boldsymbol{B}$ to uncountable models of the same cardinality.

(e) Conclude that for each cardinal $\kappa \neq 0$ (e.g. $\kappa \in \mathbb{N}^+$ or $\kappa := \aleph_0$), T is equivalent to $\mathrm{Th}(\boldsymbol{Z}_\kappa)$, where $\boldsymbol{Z}_\kappa$ is the unique (up to isomorphism) model of T that has κ-many connected components. In particular, $\mathrm{Th}(\boldsymbol{Z}_1) = \mathrm{Th}(\boldsymbol{Z}_\kappa)$ for all cardinals $\kappa \neq 0$.

REMARK: The fact that $\mathrm{Th}(\boldsymbol{Z}_1) = \mathrm{Th}(\boldsymbol{Z}_2)$ illustrates, once again, that connectedness is not captured by first-order logic.

5. Hall's marriage theorem for infinite graphs. A **matching** in an (undirected with no loops) graph $\boldsymbol{G} := (V, E)$ is a set M of (undirected) edges such that no two edges in M are adjacent. For a subset $U \subseteq V$ of vertices, a U-**perfect matching** is a matching M such that each vertex in U is incident to a (necessarily unique) edge in M. A V-perfect matching is just called a **perfect matching**. Finally, denote by $N_{\boldsymbol{G}}(U)$ the set of all vertices that have a neighbour in U.

Theorem (Hall's marriage, finite graphs)**.** *Let $\boldsymbol{G} := (V, E)$ be a finite bipartite graph with a bipartition $V := X \cup Y$. Then there is an X-perfect matching if and only if $|N_{\boldsymbol{G}}(U)| \geqslant |U|$ for each $U \subseteq X$.*

Using Hall's marriage theorem for finite graphs deduce the following version for infinite locally finite[1] graphs:

Theorem (Hall's marriage, infinite graphs)**.** *Let $\boldsymbol{G} := (V, E)$ be a locally finite bipartite graph with a bipartition $V := X \cup Y$. Then there is an X-perfect matching if and only if $|N_{\boldsymbol{G}}(U)| \geqslant |U|$ for each finite $U \subseteq X$.*

6. A **colouring** of a set X with a set K is just a function $c : X \to K$, and we refer to the elements of K as **colours**. A **finite colouring** of X is a colouring with a finite set of colours. For a colouring $c : X \to K$, a **colour class** is a set of the form $c^{-1}(k)$ for some $k \in K$.

The following is a well known theorem of additive combinatorics:

Theorem (van der Waerden, infinitary)**.** *For every finite colouring of $\mathbb{N}$, one of the colour classes contains arbitrarily long arithmetic progressions.*

Use this theorem and compactness to derive the following finitary version:

[1] Every vertex has only finitely many neighbours.

Theorem (van der Waerden, finitary). *For each $k, \ell \in \mathbb{N}^+$, there exists $n \in \mathbb{N}^+$ such that for each colouring of $\{0, 1, \ldots, n-1\}$ with k colours, one of the colour classes contains an arithmetic progression of length ℓ.*