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Mathematical Logic HOMEWORK 5 **Due: Mar 20 (Wed)**

0. Let T be a σ-theory, φ and ψ be σ-sentences. Prove:

(a) **"$\models$" in terms of satisfiability:** $T \models \varphi$ if and only if $T \cup \{\neg\varphi\}$ is not satisfiable.

(b) **Deduction:** $T \models (\varphi \to \psi)$ if and only if $T \cup \{\varphi\} \models \psi$.

(c) **Constant/exists elimination:** Let $\theta(\vec{x})$ be an extended σ-formula, where $\vec{x} := (x_1, x_2, \dots, x_n)$. Let $\vec{c} := (c_1, c_2, \dots, c_n)$ be a vector of constant symbols which do not appear in σ and let $\theta(\vec{c})$ be the sentence in the signature $\tilde{\sigma} := \sigma \cup \{c_1, c_2, \dots, c_n\}$ obtained by replacing each variable x_i in θ with c_i, for $i = 1, 2, \dots, n$. Then

$$T \cup \{\theta(\vec{c})\} \models \psi \text{ if and only if } T \cup \{\exists \vec{x} \theta(\vec{x})\} \models \psi.$$

1. Let $\boldsymbol{A}_0, \boldsymbol{A}_1, \dots$ be σ-structures such that $\boldsymbol{A}_0 \subseteq \boldsymbol{A}_1 \subseteq \boldsymbol{A}_2 \subseteq \dots$.

(a) Show that there is a unique σ-structure $\boldsymbol{A}$ with underlying set $A := \bigcup_{n \in \mathbb{N}} A_n$ such that $\boldsymbol{A}_n \subseteq \boldsymbol{A}$ for all $n \in \mathbb{N}$.

(b) Prove that $\boldsymbol{A}_0 \preceq \boldsymbol{A}_1 \preceq \boldsymbol{A}_2 \preceq \dots$ if and only if $\boldsymbol{A}_n \preceq \boldsymbol{A}$ for all $n \in \mathbb{N}$.

2. Sufficient condition for elementarity. Let $\boldsymbol{B}$ be a σ-structure and $\boldsymbol{A} \subseteq \boldsymbol{B}$. Suppose that for every finite $P \subseteq A$ and $b \in B$, there exists an automorphism h of $\boldsymbol{B}$ that fixes P pointwise (i.e. $h(p) = p$ for all $p \in P$) and sends b into A, i.e. $h(b) \in A$. Prove that $\boldsymbol{A} \preceq \boldsymbol{B}$.

3. Prove that $(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$. Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$, but $(\mathbb{Q}, <) \not\cong (\mathbb{R}, <)$.

HINT: Use Question 2 and the ultrahomogeneity of $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ (see Question 2(a) of Homework 2).

4. Let σ be a signature.

(a) Prove that if a σ-theory T is finitely axiomatizable, then there is a finite axiomatization $T_0 \subseteq T$ of T.

(b) Deduce that the class of all infinite σ-structures is not finitely axiomatizable.

(c) Also deduce that the class of all bipartite graphs is not finitely axiomatizable.

5. Lefschetz Principle (weak version). Let φ be a σ_{rng}-sentence, where $\sigma_{\text{rng}} := (0, 1, +, \cdot)$ is the signature for rings. Prove that if φ holds in all algebraically closed fields of characteristic zero (i.e. $\text{ACF}_0 \models \varphi$), then it holds in all algebraically closed fields of large enough characteristic (i.e. $\text{ACF}_p \models \varphi$ for large enough primes $p \in \mathbb{N}$).

REMARK: The strong version says that this is "if and only if", and it follows immediately from the completeness of ACF_0 and ACF_p, which we will prove later.

REMARK: Solomon Lefschetz was an algebraic topologist/geometer who stated this as a philosophical principle, and logicians turned this into an actual theorem.

6. Let σ be a signature and T, S be σ-theories. Suppose that for each σ-structure $\boldsymbol{M}$,

$$\boldsymbol{M} \models \varphi \text{ for } \textit{every } \varphi \in T \text{ if and only if } \boldsymbol{M} \models \psi \text{ for } \textit{some } \psi \in S.$$

In other words, if we allowed ourselves to write infinite conjunctions and disjunctions, we would *informally* write

$$\bigwedge_{\varphi \in T} \varphi \iff \bigvee_{\psi \in S} \psi.$$

Prove that there are finite subsets $T_0 \subseteq T$ and $S_0 \subseteq S$ such that the sentences $\bigwedge_{\varphi \in T_0} \varphi$ and $\bigvee_{\psi \in S_0} \psi$ are equivalent, i.e.

$$\emptyset \models \Big(\bigwedge_{\varphi \in T_0} \varphi \leftrightarrow \bigvee_{\psi \in S_0} \psi\Big).$$

Deduce that T_0 axiomatizes T, so T is finitely axiomatizable.

Hint: Prove that $T \cup \{\neg\psi : \psi \in S\}$ contains a finite non-satisfiable subset F, and let $T_0 := \{\varphi \in T : \varphi \in F\}$ and $S_0 := \{\psi \in S : \neg\psi \in F\}$.

7. [*Optional*] **The logic topology.** For a signature σ, let $\mathcal{T}_\sigma$ denote the set of all maximal satisfiable σ-theories and equip it with the topology generated by the sets of the form

$$[\varphi] := \{T \in \mathcal{T}_\sigma : \varphi \in T\}$$

for a σ-sentence φ.

(a) Show that the sets $[\varphi]$ are clopen and form a basis for this topology, making it a 0-dimensional Hausdorff space homeomorphic to a subset of $2^{\text{Sentences}(\sigma)}$.

(b) Realize that the Compactness theorem simply says that $\mathcal{T}_\sigma$ is compact. Thus, $\mathcal{T}_\sigma$ is homeomorphic to a *closed* subset of $2^{\text{Sentences}(\sigma)}$.

Hint: Recall the definition of compactness in terms of collections of closed sets with the finite intersection property.

(c) Observe that Question 6 simply says that the only clopen sets in $\mathcal{T}_\sigma$ are the basic ones, i.e. the sets of the form $[\varphi]$.