Last update: 2:22am (Yerevan), Mar 12.

Mathematical Logic | HOMEWORK 4 | **Due: Mar 13 (Wed)**

1. Let $\sigma \subseteq \sigma'$ be signatures. Prove that if a σ-structure $\boldsymbol{A} := (A, \sigma)$ is a reduct of a σ'-structure $\boldsymbol{A}' := (A, \sigma')$, then for every σ-formula $\varphi(\vec{v})$ and $\vec{a} \in A^n$,

$$\boldsymbol{A} \models \varphi(\vec{a}) \text{ if and only if } \boldsymbol{A}' \models \varphi(\vec{a}).$$

2. For each σ-structure $\boldsymbol{A}$ and $P \subseteq A$, the collection $\mathrm{Def}_{\boldsymbol{A}}(P)$ of P-definable sets of $\boldsymbol{A}$ is the smallest P-constructively closed collection containing

- the constant singletons: $\{c^{\boldsymbol{A}}\}$ for each $c \in \mathrm{Const}(\sigma)$;
- the graphs of functions: $\mathrm{Graph}(f^{\boldsymbol{A}})$ for each $f \in \mathrm{Func}(\sigma)$;
- the relations $R^{\boldsymbol{A}}$ for each $R \in \mathrm{Rel}(\sigma)$ and the equality relation (i.e. the diagonal in A^2).

3. Let σ be a finite signature without function symbols (e.g. the signature for graphs).

(a) Prove that for each satisfiable existential σ-sentence φ there is a finite collection $\mathcal{F}$ of finite σ-structures such that for every σ-structure $\boldsymbol{A}$, we have $\boldsymbol{A} \models \varphi$ if and only if $\boldsymbol{A}$ has a substructure isomorphic to one in $\mathcal{F}$.

(b) Deduce the dual statement for universal formulas: for each universal σ-sentence φ there is a finite collection $\mathcal{F}$ of finite σ-structures (**forbidden patterns**) such that for every σ-structure $\boldsymbol{A}$, we have $\boldsymbol{A} \models \varphi$ if and only if no substructure of $\boldsymbol{A}$ is isomorphic to one in $\mathcal{F}$.

(c) [*Optional*] Recall that a (simple) graph is called **planar** if it can be drawn on the plane without any two edges intersecting (more precisely, embedded into $\mathbb{R}^2$ as a topological space). This definition itself is not first-order, nevertheless the class of planar graphs is axiomatizable due to Kuratowski's theorem. Show that in fact, the class of planar graphs is axiomatizable by a **universal theory**, i.e. a theory containing only universal sentences.

4. Prove that for a σ-theory T the following are equivalent:

(1) T is semantically σ-complete.

(2) $\mathrm{Th}(\boldsymbol{A}) = \mathrm{Thm}_\sigma(T)$ or each σ-structure $\boldsymbol{A} \models T$, where

$$\mathrm{Thm}_\sigma(T) := \{\varphi \in \mathrm{Sentences}(\sigma) : T \models \varphi\}$$

is the set **theorems of** T.

(3) $\boldsymbol{A} \equiv \boldsymbol{B}$ for all σ-structures $\boldsymbol{A}, \boldsymbol{B} \models T$.

5. Let $n \in \mathbb{N}^+$ and abbreviate $\dot{n} := \underbrace{1 + 1 + \cdots + 1}_{n}$. Prove:

(a) For each prime $p \in \mathbb{N}$ and a natural number $n \in \mathbb{N}$, prove that

$$\text{FIELDS}_p \models \dot{n} = \dot{r},$$

where r is the remainder of the division of n by p.

(b) $\text{FIELDS}_0 \models \dot{n} \neq 0$.

6. Prove:

(a) $\text{PA} \models \forall x \forall y \forall z[(x+y)+z = x+(y+z)]$,

(b) $\text{PA} \models \forall x(0+x = x)$,

(c) $\text{PA} \models \forall x \forall y(x+y = y+x)$.

Caution: PA has **many** models different from $\boldsymbol{N}$, even uncountable ones.