Mathematical Logic — HOMEWORK 3 — **Due: Mar 6 (Wed)**

Below let σ denote a signature.

1. Prove that every set $D \subseteq \mathbb{N}^k$ that is definable in $\boldsymbol{N} := (\mathbb{N}, 0, S, +, \cdot)$ is actually $\emptyset$-definable.

2. Determine whether the following sets/elements/functions are $\emptyset$-definable in respective structures; prove all your answers.

(a) The set of non-negative numbers in $(\mathbb{Q}, +)$.

(b) The set of non-negative numbers in $(\mathbb{Q}, +, \cdot)$.

HINT: Recall Lagrange's four squares theorem for natural numbers.

(c) The function $\max(x, y)$ in $(\mathbb{R}, <)$.

(d) The function $\text{mean}(x, y) = \frac{x+y}{2}$ in $(\mathbb{R}, <)$.

(e) The element 2 in $(\mathbb{R}, +, \cdot)$.

(f) The set of prime numbers in $(\mathbb{N}, \cdot)$.

3. [*Optional*] Let $\boldsymbol{C}_{\exp} = (\mathbb{C}, 0, 1, +, \cdot, \exp)$, where exp is the usual exponentiation $z \mapsto e^z$. Show that $\mathbb{Z}$ is definable in $\boldsymbol{C}_{\exp}$. Conclude that so is $\mathbb{N}$.

4. Let $\sigma_{\text{po}} := (<)$ be the usual signature for strict partial orders. A strict linear order $(A, <)$ is called **dense** if for any $a, b \in A$ with $a < b$, there is $c \in A$ such that $a < c < b$.

(a) Write down a σ_{po}-theory DLO axiomatizing the class of all dense linear order without endpoints (i.e. without a least/largest elements).

(b) Verify that $(\mathbb{Q}, <)$ is a countable[1] model of DLO. You may use that $\mathbb{Q}$ is countable without proof.

(c) [*Optional*] Prove that $(\mathbb{Q}, <)$ is the only countable model of DLO up to isomorphism, i.e. all countable models are isomorphic $(\mathbb{Q}, <)$.

HINT: Let $\boldsymbol{A}, \boldsymbol{B}$ be two countable models, enumerate $A = \{a_n\}_{n\in\mathbb{N}}$ and $B = \{b_n\}_{n\in\mathbb{N}}$ and build an increasing sequence (f_k) of partial isomorphisms $f_k : A_k \to B_k$ where $A_k \subseteq A$ and $B_k \subseteq B$ are finite, such that $A_{2k} \supset \{a_n\}_{n\leqslant k}$ and $B_{2k+1} \supseteq \{b_n\}_{n\leqslant k}$. The method of building such a sequence of partial isomorphisms is called the **back-and-forth method**.

(d) Let $\sigma_{\text{po}}^{\infty} := (<, \{c_n : n \in \mathbb{N}\})$ and add axioms to DLO to get a theory DLO_{∞} axiomatizing the class of dense linear orders without endpoints that satisfy $c_i < c_{i+1}$ for all $i \in \mathbb{N}$.

[1] A set is countable if it is finite or admits a bijection to $\mathbb{N}$.

(e) Provide at least two nonisomorphic countable models of DLO_∞. If you can, provide three nonisomorphic models.

REMARK: In fact, DLO_∞ has exactly three nonisomorphic countable models, but you do not need to prove this.

5. Try to find an axiomatization for each of the following classes of structures. If you can find one, write it down explicitly (you don't need to prove that your axiomatization works). If you cannot find an axiomatization, just write that you don't think it's axiomatizable.

(i) Groups that contain elements of arbitrarily large finite order[2].

(ii) Groups in which every element has a finite order.

(iii) Cycle graphs[3], i.e. undirected graphs that look like an undirected cycle of some length.

(iv) Acyclic graphs, i.e. undirected graphs that do not contain any cycle.

[2]The order of a group element g is the smallest $n \in \mathbb{N}^+$ such that $g^n := \underbrace{g \cdot g \cdot \ldots \cdot g}_{n \text{ times}} = 1$, if such a positive integer exists; otherwise, the order of g is ∞.

[3]Formally, a **cycle graph** is an undirected graph $\boldsymbol{G} := (V, E^{\boldsymbol{G}})$ such that $V = \{v_0, v_1, \ldots v_n\}$ for some n, where $v_1, \ldots, v_n$ are pairwise distinct, $v_0 = v_n$, and $E^{\boldsymbol{G}} = \{(v_i, v_{i+1}), (v_{i+1}, v_i) : 0 \leqslant i < n\}$.