Last update: 11:11pm (Yerevan), Feb 25.

Mathematical Logic HOMEWORK 2 **Due: Feb 28 (Wed)**

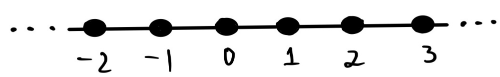
Below let σ denote a signature.

1. A σ-structure is called **rigid** if it has no automorphisms[1] other than the identity. Show that the structures $\boldsymbol{N} := (\mathbb{N}, 0, S, +, \cdot)$ and $\boldsymbol{Q} := (\mathbb{Q}, 0, 1, +, \cdot)$ are rigid. Here, the symbol S is interpreted as the successor operation $n \mapsto n+1$ and all other symbols are with their standard interpretations.

2. A σ-structure $\boldsymbol{A}$ is called **ultrahomogeneous** if any σ-isomorphism between two finitely generated[2] substructures extends to a σ-automorphism of the whole structure $\boldsymbol{A}$, i.e. if $\boldsymbol{B}, \boldsymbol{C}$ are finitely generated substructures of $\boldsymbol{A}$ and $h : \boldsymbol{B} \xrightarrow{\sim} \boldsymbol{C}$ is a σ-isomorphism, then there is a σ-automorphism $\bar{h}$ of $\boldsymbol{A}$ extending h.

(a) Show that $(\mathbb{Q}, <)$ is ultrahomogeneous. The same argument should prove that $(\mathbb{R}, <)$ is also ultrahomogeneous.

(b) Let $\sigma_{\text{gph}} := (E)$ be the signature for graphs and let $\boldsymbol{G}$ be a graph (i.e. a σ_{gph}-structure) that is the undirected[3] bi-infinite path, namely, $\boldsymbol{G} := (\mathbb{Z}, E^{\boldsymbol{G}})$, where $E^{\boldsymbol{G}} := \{(x, y) \in \mathbb{Z} : |x - y| = 1\}$. Is $\boldsymbol{G}$ ultrahomogeneous? Prove your answer.



3. Let $\boldsymbol{R} := (\mathbb{R}, 0, 1, +, \cdot)$ with the usual interpretation of the symbols. Let $t_1 := x^2 + 1$, $t_2 := \dot{2} \cdot x$, $\varphi := t_1 = t_2$, $\psi := \exists x \varphi$, $\eta := \exists y \varphi$, where $x^2 := x \cdot x$ and $\dot{2} := (1+1)$. Explicitly describe and draw the following functions and relations: $t_1^{\boldsymbol{R}}(x)$, $t_1^{\boldsymbol{R}}(x, y)$, $\varphi^{\boldsymbol{R}}(x)$, $\psi^{\boldsymbol{R}}$, $\psi^{\boldsymbol{R}}(y)$, $\eta^{\boldsymbol{R}}(x)$, and $\eta^{\boldsymbol{R}}(x, z)$, where we simply write ψ for the extended formula $\psi()$ with the empty vector of variables.

4. (a) A σ-formula is called **existential** (resp. **universal**) if it is of the form $\exists x_1 \exists x_2 ... \exists x_n \psi$ (resp. $\forall x_1 \forall x_2 ... \forall x_n \psi$) for some quantifier free σ-formula ψ. Let $\boldsymbol{B}$ be σ-structure, $\boldsymbol{A} \subseteq \boldsymbol{B}$ be a substructure, and $\varphi(\vec{v})$ be an extended σ-formula with $n := |\vec{v}|$. Show that for each $\vec{a} \in A^n$,

(i) if φ is quantifier free, then: $\boldsymbol{A} \models \varphi(\vec{a})$ if and only if $\boldsymbol{B} \models \varphi(\vec{a})$;

(ii) if φ is existential, then: $\boldsymbol{A} \models \varphi(\vec{a})$ implies $\boldsymbol{B} \models \varphi(\vec{a})$;

(iii) if φ is universal, then: $\boldsymbol{B} \models \varphi(\vec{a})$ implies $\boldsymbol{A} \models \varphi(\vec{a})$.

(b) Find a sentence that is true in $(\mathbb{N}, <)$ but false in $(\mathbb{Z}, <)$, and vice versa.

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[1]An **automorphism** of a σ-structure $\boldsymbol{A}$ is just an isomorphism from $\boldsymbol{A}$ to itself.

[2]**Finitely generated** means generated by a finite subset.

[3]By an **undirected** graph we mean that E is a symmetric relation.

5. [*Optional*] Let $\vec{x_1}, ..., \vec{x_n} \in \mathbb{Q}^m$. Show that $\{\vec{x_1}, ..., \vec{x_n}\}$ is linearly independent over $\mathbb{Q}$ if and only if it is linearly independent over $\mathbb{R}$.

   Hint: Show that linear independence can be expressed by both universal and existential formulas, or by simply a quantifier free formula.

6. Let $\sigma := (f)$ be a signature, where f is a unary function symbol. Find a σ-sentence φ such that there are σ-structures satisfying φ and all σ-structures satisfying φ are infinite.

7. Let σ be a signature and $n \in \mathbb{N}$. Explicitly write down a σ-sentence φ_n such that the σ-structures satisfying φ_n are exactly those that have n elements.

8. Let σ be a *finite* signature and $\boldsymbol{A}$ be a *finite* σ-structure. Show that there is a σ-sentence φ that uniquely determines $\boldsymbol{A}$ up to isomorphism, i.e. for every σ-structure $\boldsymbol{B}$,

$$\boldsymbol{B} \models \varphi \text{ if and only if } \boldsymbol{B} \cong \boldsymbol{A}.$$

   Hint: If $n := |A|$, then φ is of the form $\exists v_1 \exists v_2 \ldots \exists v_n \psi$, just like the formula φ_n from Question 7, but ψ also describes the interpretations of constant, function, and relations symbols in $\boldsymbol{A}$, e.g. it specifies which relations hold between the elements $v_1, v_2, \ldots, v_n$. Do this first with a finite group (e.g. $\mathbb{Z}/3\mathbb{Z}$) to understand concretely what's going on.