Math Logic: Model Theory & Computability
Lecture 15

Classes defined by a quantifier followed by an infinite conjunction.

Examples. Just take the complements of our previous examples with a quantifier plus an infinite disjunction:

(a) The class of non-torsion groups, i.e. groups $\underline{G}$ such that
$$\exists g \bigwedge_{n \in \mathbb{N}^+} \underbrace{g \cdot g \cdot \ldots \cdot g}_{n} \neq 1_{\underline{G}}.$$

(b) The class of disconnected graphs: $\underline{G} := (V, E)$ s.t.
$$\exists u \exists v \bigwedge_{n \in \mathbb{N}^+} \neg \varphi_n(u, v),$$
where $\varphi_n(x, y)$ says that there is a path in $\underline{G}$ of length n between x and y.

We prove that (b) is not axiomatizable, leaving the nonaxiomatizability of (a) as an exercise.

Prop. The class of disconnected graphs is not axiomatizable.

Proof. Suppose toward a contradiction that there is a $\sigma_{gph} := (E)$-theory T axiomatizing this class. Let $\tilde{\sigma} := \sigma_{gph} \cup \{a, b\}$, where a, b are const. symbols and define
$$S := \{ d_{>n}(a, b) : n \in \mathbb{N} \}.$$
Because the reduct of every model $\underline{M}$ of S to σ_{gph} is disconnected, $\underline{M} \models T$. This means that for each sentence $\varphi \in T$, $S \models \varphi$. We know by compactness that $S_\varphi \models \varphi$ for some finite subset $S_\varphi \subseteq S$. Because $d_{>n}(a, b) \models d_{>m}(a, b)$ for all $m \leq n$, we may assume that $S_\varphi = \{ d_{>n}(a, b) \}$ for some $n \in \mathbb{N}$. Thus,

$d_{>n}(a,b) \models \varphi$, where $n := n_\varphi$. By constant elimination (HW 5, Q0), this is equivalent to

$$\exists x \exists y \, d_{>n}(x,y) \models \varphi.$$

But then the theory $S' := \{\exists x \exists y \, d_{>n}(x,y) : n \in \mathbb{N}\} \models \varphi$ for all $\varphi \in T$, i.e. $S' \models T$, which is a contradiction because, say, a $\mathbb{Z}$-like graph (2-regular acyclic) satisfies S' but isn't disconnected, hence shouldn't satisfy T. $\square$

Positive applications of compactness.

From finite to infinite.

The nature of compactness is boosting finite to infinite. For example:

Cor. If a theory T has arbitrarily large finite models, then it has an infinite.

Another example is graph colouring problems or any other locally checkable problem.

Theorem (De Bruijn–Erdős). Let $k \geq 2$. If every finite subgraph of a given graph $\underline{G} := (V,E)$ is k-colourable (admits a proper vertex colouring with k colours), then $\underline{G}$ is k-colourable.

Proof. For notational convenience, we prove for $k=3$, but the idea of the proof is the same for all k. Let $\tilde{\sigma} := \sigma_{gph} \cup \{R_1, R_2, R_3\}$ where the R_i are unary relation symbols (to be interpreted as colours). Let φ be a $\tilde{\sigma}$-sentence stating that R_1, R_2, R_3 forms a proper colouring, e.g. φ is the conjunction of:

(i) $\forall x \, (R_1(x) \vee R_2(x) \vee R_3(x))$

(ii) $\forall x \left[R_1(x) \to (\neg R_2(x) \wedge \neg R_3(x)) \right] \wedge \left[R_2(x) \to (\neg R_1(x) \wedge \neg R_3(x)) \right] \wedge$
$\wedge \left[R_3(x) \to (\neg R_1(x) \wedge \neg R_2(x)) \right]$

(iii) $\forall x \forall y \left[x E y \to \bigwedge_{i=1}^{3} \neg (R_i(x) \wedge R_i(y)) \right]$.

Let $T = \{\varphi\} \cup \{c_u E c_v : u, v \in V \text{ such that } u E^G v\}$. Then every finite $T_0 \subseteq T$ is satisfiable by our hypothesis: indeed, let $c_{v_1}, \dots, c_{v_n}$ be all constants that appear in T_0, then the finite induced subgraph $\underline{H}$ on vertices $\{v_1, \dots, v_n\}$ admits a 3-colouring, by our hypothesis, so $\underline{H}$ admits an expansion to a σ-theory satisfying T_0. Thus T has a model, i.e. a σ-structure $\underline{M}$ where $\sigma = \{E, R_1, R_2, R_3\} \cup \{c_v : v \in V\}$. The reduct of $\underline{M}$ to a σ_{gph}-structure is a 3-colourable graph s.t. $\underline{G}$ injectively homomorphs into it by the map $v \mapsto c_v^{\underline{M}}$. Thus, $\underline{G}$ too is 3-colourable being a subgraph of a 3-colourable graph. $\square$

From infinite to finite.

Because the nature of the compactness theorem is from finite to infinite, we should use its contrapositive to get from infinite to finite: if a theory T does not have a model then some finite subtheory of T doesn't have a model. These are called compactness-and-contradiction arguments.

We illustrate this on the example of Ramsey's theorem, obtaining the finitary version from its infinitary version (which is much easier to prove).

Def. For a set X and $\ell \in \mathbb{N}^+$, let $[X]^\ell$ denote the set of all ℓ-element subsets of X. So $[X]^2$ is the set of all undirected edges between the elements of X, while $[X]^\ell$ is the set of all ℓ-hyperedges between the elements of X.

A colouring of $[X]^\ell$ with k colours is just a function $c : [X]^\ell \to \{0, \dots, k-1\}$.

A subset $E \subseteq [X]^\ell$ is called c-monochromatic if $c|_E$ is constant.
A subset $X' \subseteq X$ is called c-monochromatic if $[X']^\ell$ is c-monochromatic.

Infinite Ramsey Theorem. For any $\overset{2}{\ell}, \overset{2}{k}$, and any colouring $c : [\mathbb{N}]^{\overset{2}{\ell}} \to \{0, \dots, k-1\}$, there is an infinite $M \subseteq \mathbb{N}$ c-monochromatic subset.

Before proving let's understand the statement on an example:

Example. Let $(R, <)$ be a linear order. Then any sequence $(r_n)_{n \in \mathbb{N}} \subseteq R$ has a monotone (increasing or decreasing, but not necessarily strictly) subsequence.

Proof. Colour a pair $i<j$ blue if $r_i \leq r_j$, otherwise colour the pair $i<j$ red. By Ramsey, $\exists$ infinite $I \subseteq \mathbb{N}$ s.t. all pairs $i<j$ in I are red or all pairs in I are blue. Then $(r_i)_{i \in I}$ is monotone. $\square$