Math Logic: Model Theory & Computability
Lecture 13

Upward Löwenheim–Skolem theorems.

Recall that by the weak downward L–S, every satisfiable σ-theory T has a model of cardinality $\leq \max(|\sigma|, \aleph_0)$. We now prove the upward version:

Weak upward Löwenheim–Skolem theorem. For a σ-theory T, the following are equivalent:

(1) For each $n \in \mathbb{N}$, T has a model of cardinality $\geq n$.
(2) For each cardinal $\kappa \geq \max(|\sigma|, \aleph_0)$, T has a model of cardinality $= \kappa$.
(3) T has an infinite model.

Proof. (2)$\Rightarrow$(3)$\Rightarrow$(1) is trivial, so we prove (1)$\Rightarrow$(2). Suppose (1) and fix an infinite cardinal $\kappa \geq |\sigma|$. Let $\tilde{\sigma} := \sigma \cup \{c_\alpha : \alpha \in \kappa\}$ so that the c_α are new constants and define

$$\tilde{T} := T \cup \{c_\alpha \neq c_\beta : \alpha, \beta \in \kappa \text{ distinct}\}.$$

Then $\tilde{T}$ is finitely satisfiable because any finite subtheory $\tilde{T}_0 \subseteq \tilde{T}$ will only contain finitely many sentences of the form $c_\alpha \neq c_\beta$ and there is a model of T with at least that many distinct elements.
By compactness, $\tilde{T}$ has a model $\tilde{\underline{B}}$. By the downward L–S, there is an elementary substructure $\tilde{\underline{A}} \preccurlyeq \tilde{\underline{B}}$ of cardinality $\leq \max(|\tilde{\sigma}|, \aleph_0) = \kappa$, hence $|\tilde{A}| = \kappa$ because $\tilde{A} \supseteq \{c_\alpha^{\tilde{\underline{A}}} : \alpha \in \kappa\}$ and the latter has cardinality κ. By elementarity, $\tilde{\underline{A}} \models T$, hence its reduct $\underline{A}$ to the σ-structure is still a model of T and still has cardinality κ. $\square$

Cor. If a σ-theory T admits arbitrarily large finite models then it admits an infinite model.

Cor. Let $\mathcal{C}$ be a class of σ-structures containing an infinite structure. If the cardinality of all structures in $\mathcal{C}$ is bounded above then $\mathcal{C}$ is not axiomatizable.

Examples. (a) The class of cyclic (i.e. 1-generated) groups is not axiomatizable because they are all countable and $(\mathbb{Z}, 0, +)$ is infinite and cyclic.
(b) The class of finitely generated groups is not axiomatizable because they are all countable and $(\mathbb{Z}, 0, +)$ is infinite and finitely generated.

Note that for a given σ-structure $\underline{A}$, by taking $T := \mathrm{Th}(\underline{A})$, we can get a σ-structure $\underline{B}$ of cardinality κ that is elementarily equivalent to $\underline{A}$. However, to make this $\underline{B}$ an elementary extension of $\underline{A}$, i.e. $\underline{B} \succcurlyeq \underline{A}$, we need $\underline{B}$ to satisfy more than just $\mathrm{Th}(\underline{A})$, namely, the elementary diagram of $\underline{A}$...

Def. Let $\underline{A}$ be a σ-structure and extend the signature σ by adding one new constant for each element of A:
$$\sigma_A := \sigma \cup \{c_a : a \in A\},$$
where the c_a are constant symbols not appearing in σ. The natural expansion of $\underline{A}$ to a σ_A-structure is the expansion $\underline{\tilde{A}} := (\underline{A}, \sigma_A)$ where $c_a^{\underline{\tilde{A}}} := a$ for all $a \in A$.
The elementary diagram of $\underline{A}$ is $\mathrm{ElDiag}(\underline{A}) := \mathrm{Th}(\underline{\tilde{A}})$, in other words, for each extended σ-formula $\varphi(\vec{x})$ and $\vec{a} := (a_1, a_2, \dots, a_k) \in A^k$ where $k := |\vec{x}|$,
$$\varphi(c_{\vec{a}}) \in \mathrm{ElDiag}(\underline{A}) \text{ iff } \underline{A} \models \varphi(\vec{a}),$$
where $c_{\vec{a}} := (c_{a_1}, c_{a_2}, \dots, c_{a_k})$.
We also denote by $\mathrm{Diag}(\underline{A})$ the subset of $\mathrm{ElDiag}(\underline{A})$ of all quantifier free σ_A-sentences, and call this the diagram of $\underline{A}$ (or the quantifier free diagram of $\underline{A}$).

Lemma. Let $\underline{A}, \underline{B}$ be σ-structures. If $\underline{B}$ admits an expansion $\underline{\tilde{B}}$ to a σ_A-structure that is a model of $\mathrm{ElDiag}(\underline{A})$, then $\underline{A} \hookrightarrow_e \underline{B}$. In particular, there is an isomorphic copy of $\underline{B}$, denote it by $\underline{B}'$, such that $\underline{A} \preceq \underline{B}'$.

Proof. Let $\tilde{\underline{B}} \models \mathrm{ElDiag}(\underline{A})$. Then the function $h: A \to B$ given by $a \mapsto c_a^{\tilde{\underline{B}}}$ is an elementary embedding of $\underline{A}$ into $\underline{B}$ because for each extended σ-formula $\varphi(\vec{x})$ and $\vec{a} \in A^{|\vec{x}|}$, we have that

$$\underline{A} \models \varphi(\vec{a}) \iff \varphi(c_{\vec{a}}) \in \mathrm{ElDiag}(\underline{A}) \iff \tilde{\underline{B}} \models \varphi(c_{\vec{a}}) \iff \underline{B} \models \varphi(h(\vec{a})),$$

where $c_{\vec{a}} := (c_{a_1}, c_{a_2}, \dots, c_{a_k})$ if $\vec{a} = (a_1, a_2, \dots, a_k)$.
Replacing $h(A)$ inside $\underline{B}$ with A, we get an isomorphic copy $\underline{B}'$ of $\underline{B}$ but now $\underline{A} \preceq \underline{B}'$. $\square$

Upward Löwenheim-Skolem Theorem. For every infinite σ-structure $\underline{A}$ and cardinal $\kappa \geq \max(|A|, |\sigma|)$, there is an elementary extension $\underline{B}$ of $\underline{A}$ of cardinality κ.

Proof. By the weak upward L-S applied to the σ_A-theory $\mathrm{ElDiag}(\underline{A})$, there is a model $\tilde{\underline{B}} \models \mathrm{ElDiag}(\underline{A})$ of cardinality κ, thus its reduct $\underline{B}$ to a σ-structure satisfies $\underline{A} \hookrightarrow_e \underline{B}$ by the above lemma. Hence there is $\underline{B}' \cong \underline{B}$ s.t. $\underline{A} \preceq \underline{B}'$. $\square$