Math Logic: Model Theory & Computability
Lecture 12

Downward Löwenheim–Skolem (uses Axiom of Choice). For each σ-structure $\underline{B}$ and $S \subseteq B$, there is $\underline{A} \preceq \underline{B}$ containing S such that $|A| \leq \max(\aleph_0, |\langle S \rangle_{\underline{B}}|) \leq \max(|S|, |\sigma|, \aleph_0)$.

A conceptual way to prove this, uses the following concept:

Def. Let $\underline{B}$ be a σ-structure and $\varphi(\vec{x}, y)$ be an extended σ-formula. A **Skolem function for $\varphi(\vec{x}, y)$** is a function $S_\varphi : B^{|\vec{x}|} \to B$ such that for each $\vec{b} \in B^{|\vec{x}|}$, if $\underline{B} \models \exists y \varphi(\vec{b}, y)$ then $\underline{B} \models \varphi(\vec{b}, S_\varphi(\vec{b}))$. In other words, $S_\varphi(\vec{b})$ is a choice of a witness for $\exists y \varphi(\vec{b}, y)$ if it holds.
A **Skolemization** of $\underline{B}$ is an expansion $\underline{\tilde{B}}$ to a $\tilde{\sigma}$-structure, where $\tilde{\sigma} := \sigma \cup \{ f_{\varphi(\vec{x}, y)} : \varphi(\vec{x}, y) \text{ is an extended } \sigma\text{-formula} \}$ and $f_{\varphi(\vec{x}, y)}$ has arity $|\vec{x}|$, and the interpretation of $f_{\varphi(\vec{x}, y)}$ in $\underline{\tilde{B}}$ is a Skolem function for $\varphi(\vec{x}, y)$.

Lemma. Let $\underline{B}$ be a σ-structure and let $\underline{\tilde{B}}$ be a Skolemization of $\underline{B}$ (in the Skolemized signature $\tilde{\sigma}$ as in the above def).
The reduct $\underline{A}$ of any substructure $\underline{\tilde{A}} \subseteq \underline{\tilde{B}}$ to a σ-structure is an elementary substructure of $\underline{B}$.

Proof. Let $\underline{A}$ be as described and check that it satisfies the Tarski–Vaught test: for each extended σ-formula $\varphi(\vec{x}, y)$ and $\vec{a} \in A^{|\vec{x}|}$, if $\underline{B} \models \exists y \varphi(\vec{a}, y)$ then $\underline{B} \models \varphi(\vec{a}, f_{\varphi(\vec{x}, y)}(\vec{a}))$ for the corresponding Skolem function symbol $f_{\varphi(\vec{x}, y)}$ in $\tilde{\sigma}$. But because $\underline{A}$ is a reduct of a $\tilde{\sigma}$-substructure of $\underline{\tilde{B}}$, A is closed under all functions of $\underline{\tilde{B}}$, in particular, $f_{\varphi(\vec{x}, y)}(\vec{a}) \in A$. Thus, $\underline{A}$ passes the Tarski–Vaught test. $\square$

Downward Löwenheim–Skolem (uses Axiom of Choice). For each σ-structure $\underline{B}$ and $S \subseteq B$, there is $\underline{A} \preccurlyeq \underline{B}$ containing S such that $|A| \leq \max(|S|, |\sigma|, \aleph_0)$.

Proof. Let $\underline{\tilde{B}}$ be a Skolemization of $\underline{B}$ (this uses axiom of choice to define each Skolem function). Let $\underline{\tilde{A}}$ be the substructure of $\underline{\tilde{B}}$ generated by S, and let $\underline{A}$ be the reduct of $\underline{\tilde{A}}$ to a σ-structure. Then $A \supseteq S$ and $\underline{A} \preccurlyeq \underline{B}$ by the previous lemma. Moreover,

$$|A| = |\langle S \rangle_{\underline{\tilde{B}}}| \leq \max(|S|, |\tilde{\sigma}|, \aleph_0)$$

but $|\tilde{\sigma}| \leq |\sigma| + |\mathrm{Formulas}(\sigma)|$ and $|\mathrm{Formulas}(\sigma)| \leq |\bigcup_{n\in\mathbb{N}} \mathrm{Alph}(\sigma)^n| \leq \max(|\sigma|, \aleph_0)$, so $|\tilde{\sigma}| \leq |\sigma| + \max(|\sigma|, \aleph_0) = \max(|\sigma|, \aleph_0)$, hence $|A| \leq \max(|S|, |\sigma|, \aleph_0)$. $\square$

Exercise. In fact, $A = \bigcup_{n\in\mathbb{N}} S_n$, where $S_0 := S$ and $S_{n+1} := \bigcup_{\varphi(\vec{x}, y)} f^{\underline{\tilde{B}}}(S_n^{|\vec{x}|})$.

Cor (weak downward Löwenheim–Skolem). If a σ-theory T is satisfiable, then it has a model of cardinality $\leq \max(|\sigma|, \aleph_0)$ (hence ctbl if σ is ctbl).

Proof. Let $\underline{B} \models T$ and let $\underline{A} \preccurlyeq \underline{B}$ be an elementary substructure (containing $S := \emptyset$) of cardinality $\leq \max(|\sigma|, \aleph_0)$. $\square$

Examples. (a) Let $\underline{R} := (\mathbb{R}, 0, 1, +, \cdot, <)$ then $\mathrm{Th}(\underline{R})$ has ctbl models. It follows from Tarski's quantifier elimination theorem for $\underline{R}$ that the substructure of algebraic reals is a ctbl model of $\mathrm{Th}(\underline{N})$.

(b) **Skolem "paradox".** If ZFC is satisfiable, then it has ctbl models.

The Weak Downward Löwenheim–Skolem theorem has the following at first striking consequence: if ZFC is satisfiable (which we really hope it is), then it has a countable model. This may seem strange because this countable model $\boldsymbol{M}$ satisfies the sentence that there is an uncountable set since Cantor's theorem that the reals are uncountable is true in $\boldsymbol{M}$. Does this imply that ZFC is not satisfiable? Of course not and here are the two reasons why (second being the main reason).

(1) Replacing the universe of M with $\mathbb{N}$, we may assume that $M = \mathbb{N}$, so ϵ^M is just a binary relation on $\mathbb{N}$, i.e. a subset of $\mathbb{N}^2$. So what if somehow $\boldsymbol{M}$ satisfies the statement that reads as "there is an uncountable set"? It is just some statement about this binary relation ϵ^M and it does not imply anything about the actual sets and the cardinality of M.

(2) Even if M was a set of sets and ϵ^M was the true $\in$, then the countability of M would simply imply that $\boldsymbol{M}$'s version of the real numbers, $\mathbb{R}^M$, is indeed countable (for us), i.e. there is a bijection $f : \mathbb{R}^M \to \mathbb{N}$. This bijection is a set, namely a subset of $\mathbb{R}^M \times \mathbb{N}$, but it may not be an element of M—the latter doesn't contain *all* sets, only countably-many of them. In fact, since $\boldsymbol{M}$ satisfies the statement "$\mathbb{R}^M$ is uncountable", we conclude that $f \notin M$ for sure! In other words, M does not "see" the countability of $\mathbb{R}^M$ and thus thinks that $\mathbb{R}^M$ is uncountable. It's like how people thought the world was endless before they discovered it was round since all they could see was the ocean up to the line of the horizon and for all they knew it continued forever. The only difference is that we eventually obtained the knowledge that Earth is round and finite, while $\boldsymbol{M}$ never will.

What about the converse to downward Löwenheim–Skolem or even just its corollary. Given a σ-structure $\underline{A}$ is there an elementary extension $\underline{B} \succeq \underline{A}$ of higher cardinality than $|A|$? In particular, does a satisfiable theory have models of arbitrarily large cardinality?
All these questions and more are answered by the most useful theorem of logic, namely, the Compactness theorem.

Compactness Theorem and its countless applications.

Def. A σ-theory T is called *finitely satisfiable* if every finite subtheory $T_0 \subseteq T$ is satisfiable (i.e. has a model).

Compactness theorem (Mal'cev, Gödel). Every finitely satisfiable σ-theory T is satisfiable.

Mal'cev proved this purely model-theoretically, using ultraproducts, and Gödel proved this as a consequence of his Completeness theorem (= semantic-syntactic duality).

Cor 1. Let T be a σ-theory and φ be a σ-sentence. If $T \models \varphi$ then $T_0 \models \varphi$ for some finite subtheory $T_0 \subseteq T$.

Proof. We prove the contrapositive: suppose $T_0 \not\models \varphi$ for all finite $T_0 \subseteq T$. Then $\{\neg\varphi\} \cup T$ is finitely satisfiable: indeed, for every finite T_0, there is a model of $\{\neg\varphi\} \cup T_0$ because $T_0 \not\models \varphi$. By compactness, $\{\neg\varphi\} \cup T$ has a model, so $T \not\models \varphi$. $\square$

Cor 2 (from Cor 1). Every finitely axiomatizable theory T admits a finite axiomatization $T_0 \subseteq T$.

Proof. HW.

Examples. (a) The theory $T_\infty := \{\exists x_1 \ldots \exists x_n \left(\bigwedge_{1\le i<j\le n} x_i \neq x_j\right) : n \in \mathbb{N}^+\}$ isn't finitely axiomatizable.

(b) The class of bipartite graphs isn't finitely axiomatizable.

HW