Math Logic: Model Theory & Computability
Lecture 11

Elementarity.

We know that for σ-structures $\underline{A} \subseteq \underline{B}$, we have for each extended σ-formula $\varphi(\vec{x})$ and $\vec{a} \in A^{|\vec{x}|}$,

(a) if φ is quantifier free, then $\underline{A} \models \varphi(\vec{a})$ $\underbrace{\text{iff}}_{\text{if and only if}}$ $\underline{B} \models \varphi(\vec{a})$.

(b) if φ is universal, then $\underline{B} \models \varphi(\vec{a})$ implies $\underline{A} \models \varphi(\vec{a})$.

(c) if φ is existensial, then $\underline{A} \models \varphi(\vec{a})$ implies $\underline{B} \models \varphi(\vec{a})$.

For general formulas, the if and only if doesn't hold for an arbitrary substructure $\underline{A}$ of $\underline{B}$.

Def. Let $\underline{A}, \underline{B}$ be σ-structures.

- A function $h : A \to B$ is called an *elementary embedding* if for each extended σ-formula $\varphi(\vec{x})$ and $\vec{a} \in A^{|\vec{x}|}$, we have
$$\underline{A} \models \varphi(\vec{a}) \quad \text{iff} \quad \underline{B} \models \varphi(h(\vec{a})).$$
We denote this by $h : \underline{A} \hookrightarrow_e \underline{B}$.
Remark. An elementary embedding is in particular an embedding because for injectivity if $h(a_0) = h(a_1)$ then $\underline{B} \models (x_0 = x_1)(h(a_0), h(a_1))$ so $\underline{A} \models (x_0 = x_1)(a_0, a_1)$, i.e. $a_0 = a_1$. Similarly for any relation $R \in \mathrm{Rel}(\sigma)$, if $\underline{B} \models R(h(\vec{a}))$, then $\underline{A} \models R(\vec{a})$. Same for function symbols $f \in \mathrm{Func}(\sigma)$.
- We say that $\underline{A}$ *elementarily embeds* into $\underline{B}$ if there is an elementary embedding $h : \underline{A} \hookrightarrow_e \underline{B}$, and this is denoted by $\underline{A} \hookrightarrow_e \underline{B}$.
- A substructure $\underline{A}$ of $\underline{B}$ is called *elementary* if the inclusion map $A \to B$, $a \mapsto a$ is an elementary embedding, in other words for each extended

σ-formula $\varphi(\vec{x})$ and $\vec{a} \in A^{|\vec{x}|}$, we have

$$\underline{A} \models \varphi(\vec{a}) \quad \text{iff} \quad \underline{B} \models \varphi(\vec{a}).$$

We denote this $\underline{A} \preccurlyeq \underline{B}$.

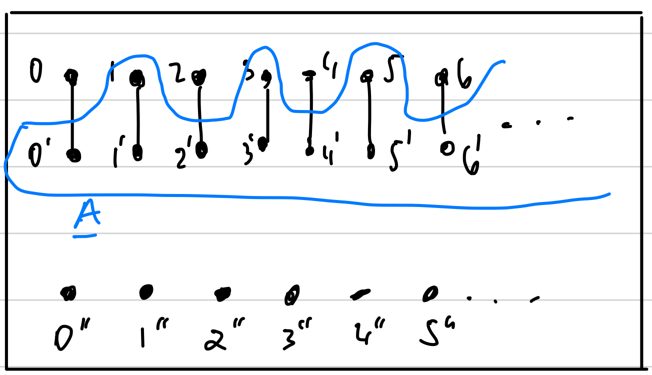
<u>(Counter) Examples.</u> (a) $(\mathbb{N}, <) \subseteq (\mathbb{Z}, <)$ but not elementary because the sentence $\forall x \exists y \, (y < x)$ is true in $(\mathbb{Z}, <)$ but false in $(\mathbb{N}, <)$.

(b) $(\mathbb{N}, 0, +) \subseteq (\mathbb{Z}, 0, +)$ but not elementary because $\forall x \exists y \, (x + y = 0)$ holds in $(\mathbb{Z}, 0, +)$ but failes in $(\mathbb{N}, 0, +)$, in other words $(\mathbb{Z}, 0, +)$ is a group but $(\mathbb{N}, 0, +)$ isn't.

(c) Unlike (a), $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$ and the proof is outlined in HW.

(d) It is clear that if $\underline{A} \preccurlyeq \underline{B}$ then in particular $\underline{A} \equiv \underline{B}$ and in examples (a) and (b) really $\equiv$ failed and hence $\preccurlyeq$ failed. We now give an example of $\underline{A} \subseteq \underline{B}$ s.t. $\underline{A} \equiv \underline{B}$ but $\underline{A} \not\preccurlyeq \underline{B}$. In fact, our $\underline{A} \subseteq \underline{B}$ will actually be isomorphic.

Let $\sigma := \sigma_{graph} := (E)$ and let $\underline{A}$ and $\underline{B}$ be the following graphs:



$\underline{B}$

Then $h : \underline{A} \xrightarrow{\cong} \underline{B}$ so in particular $\underline{A} \equiv \underline{B}$.

$$\begin{cases} (2k)' \mapsto k'' \\ (2k+1)' \mapsto k' \\ (2k+1) \mapsto k \end{cases}$$

However, $\underline{A} \not\preccurlyeq \underline{B}$ because in $\underline{B}$ $0'$ has a neighbour but it is isolated in $\underline{A}$, i.e. $\underline{B} \models \varphi(0')$ but $\underline{A} \not\models \varphi(0')$, where $\varphi(x) := \exists y \, (x E y)$.

(e) Similarly, $\underline{A} := (2\mathbb{Z}, <)$ is a substructure of $\underline{B} := (\mathbb{Z}, <)$ such that $\underline{A} \xrightarrow[2k \mapsto k]{\cong} \underline{B}$ hence $\underline{A} \equiv \underline{B}$ but $\underline{A} \not\preccurlyeq \underline{B}$ because in $\underline{B}$ there is an element between 0 and 2 but there isn't one in $\underline{A}$.

Now we give a criterion/test for a substructure to be elementary.

Tarski–Vaught test (for elementarity). For a σ-structure $\underline{B}$, a substructure $\underline{A} \subseteq \underline{B}$ is elementary iff for every extended σ-formula $\varphi(\vec{x}, y)$ and $\vec{a} \in A^{|\vec{x}|}$, if $\underline{B} \models \exists_y \varphi(\vec{a}, y)$ then there is a (Tarski–Vaught witness) $a' \in A$ such that $\underline{B} \models \varphi(\vec{a}, a')$.

Proof. $\Rightarrow$. Suppose $\underline{A} \preccurlyeq \underline{B}$ and $\underline{B} \models \exists_y \varphi(\vec{a}, y)$. By elementarity, $\underline{A} \models \exists_y \varphi(\vec{a}, y)$, so there is $a' \in A$ such that $\underline{A} \models \varphi(\vec{a}, a')$. By elementarity again, $\underline{B} \models \varphi(\vec{a}, a')$.

$\Leftarrow$. Suppose the Tarski–Vaught condition holds. To prove $\underline{A} \preccurlyeq \underline{B}$ we show by induction on formulas that for each extended formula $\varphi(\vec{x})$ and $\vec{a} \in A^{|\vec{x}|}$, we have $\underline{A} \models \varphi(\vec{a})$ iff $\underline{B} \models \varphi(\vec{a})$.

Case 1. φ is atomic, i.e. either $t_1 = t_2$ or $R(t_1, \dots, t_k)$, for σ-terms $t_1, \dots, t_k$. Then φ is quantifier free so $\underline{A} \models \varphi(\vec{a})$ iff $\underline{B} \models \varphi(\vec{a})$.

Case 2. $\varphi := \neg \psi$. Then
$$\begin{aligned} \underline{A} \models \neg\psi(\vec{a}) &\text{ iff } \underline{A} \not\models \psi(\vec{a}) \\ \text{(by induction) } &\text{ iff } \underline{B} \not\models \psi(\vec{a}) \\ &\text{ iff } \underline{B} \models \neg\psi(\vec{a}). \end{aligned}$$

Case 3. $\varphi := \psi_1 \vee \psi_2$. Similar to Case 2.

(⇒ by the Tarski–Vaught condition)

Case 4. $\varphi(\vec{x}) := \exists_y \psi(\vec{x}, y)$. Then
$$\begin{aligned} \underline{B} \models \exists_y \psi(\vec{a}, y) &\text{ iff there is } a' \in A \text{ s.t. } \underline{B} \models \psi(\vec{a}, a') \\ \text{(by induction) } &\text{ iff there is } a' \in A \text{ s.t. } \underline{A} \models \psi(\vec{a}, a') \\ &\text{ iff } \underline{A} \models \exists_y \psi(\vec{a}, y). \end{aligned}$$
$\square$

Thus a substructure is elementary if it contains a Tarski–Vaught witness for each formula of the form $\exists y\, \varphi(\vec{x}, y)$. Recall that we could define the substructure generated by a subset because intersection of substructures is again a substructure. This is not true for elementary substructures:

Example. Let $\sigma := \emptyset$. Let $\underline{B} := (\mathbb{Z})$, $\underline{A}_- := (-\mathbb{N})$ and $\underline{A}_+ := (\mathbb{N})$.
Then it is easy to check that $\underline{A}_-, \underline{A}_+ \preceq \underline{B}$ (by the Tarski-Vaught test or by the criterion via automorphism given in HW5) but $(-\mathbb{N}) \cap \mathbb{N} = \{0\}$ and $\underline{A}_0 := (\{0\})$ is not an elementary substructure of $\underline{B}$ because $\underline{A}_0 \not\equiv \underline{B}$ since $\varphi := \exists x_1 \exists x_2 (x_1 \neq x_2)$ holds in $\underline{B}$ but fails in $\underline{A}_0$.

Thus, for a given subset $S \subseteq B$ we can't define "the smallest elementary substructure of $\underline{B}$ containing S". However, we can still find an elementary substructure $\underline{A} \preceq \underline{B}$ containing S that has as small as possible cardinality, namely, $|A| \leq \max(|\langle S \rangle_{\underline{B}}|, \aleph_0)$.

Downward Löwenheim-Skolem (uses Axiom of Choice). For each σ-structure $\underline{B}$ and $S \subseteq B$, there is $\underline{A} \preceq \underline{B}$ containing S such that $|A| \leq \max(|S|, |\sigma|, \aleph_0)$.