Math Logic: Model Theory & Computability
Lecture 09

Examples (continued). (b) The theory of $\overbrace{\text{undirected graphs without loops}}^{\text{simple}}$ in the signature of graphs $\sigma_{\text{grph}} := (E)$ is UGRAPHS:

(i) $\forall x \forall y \, (x E y \to y E x)$

(ii) $\forall x \, (\neg x E x)$

(c) Recall that a (simple) graph $G := (V, E)$ is called bipartite if V admits a partition $V = V_1 \cup V_2$ such that there are no edges between the vertices in V_i, for $i = 1, 2$. Equivalently, G admits a proper colouring with 2 colours: 1 and 2. (A proper coloring of a graph with n colours is a function $c : V \to \bar{n} := \{0, 1, \dots, n-1\}$ such that adjacent verticies get different colours.)

Is the class of bipartite graphs axiomatizable? Just from the definition it seems like not because we would need to express "there is a subset $V_1 \subseteq V$ such that blablabla...". However, there is an equivalent condition to bipartiteness that is first-order expressible:

Prop. A graph is bipartite if and only if it has no odd cycles.

Proof. $\Rightarrow$. Straightforward because odd cycles are not 2-colourable.

$\Leftarrow$. If no odd cycles, we can colour the graph as follows: choose a starting point from each component, colour it red, then its neighbours blue, then their new neighbours red, and so on. We will never reach a situation where a vertex is a neighbour of both a blue and a red vertex because this implies an odd cycle. $\square$

Using this, the theory BIPGRAPHS := UGRAPHS $\cup \{\neg \varphi_{2k+1} : k \in \mathbb{N}^+\}$ axiomatizes the class of bipartite graphs, where for $n \geq 2$, φ_n says that there is a simple (no repeated vertices) cycle of length n:

$$\varphi_n := \exists x_0 \exists x_1 \ldots \exists x_n \left[\left(\bigwedge_{i=0}^{n} x_i \, E \, x_{i+1} \right) \wedge (x_0 = x_n) \wedge \left(\bigwedge_{0 \leq i < j < n} x_i \neq x_j \right) \right].$$

(c) In the signature $\sigma_{po} := (\leq)$, the theory PO:

(PO1) $\forall x \, (x \leq x)$

(PO2) $\forall x \forall y \, ((x \leq y \wedge y \leq x) \rightarrow x = y)$

(PO3) $\forall x \forall y \forall z \, ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$

axiomatizes the class of partial orders. Adding

(LO) $\forall x \forall y \, (x \leq y \vee y \leq x)$

axiomatizes the class of linear orders. However, there is no axiomatization for the class of well-orders, namely those linear orders in which every nonempty subset has a minimum.

(d) In $\sigma_{grp} := (1, \cdot, ()^{-1})$, the theory axiomatizing all groups is GROUPS:

(GP1) $\forall x \forall y \forall z \, ((x \cdot y) \cdot z = x \cdot (y \cdot z))$

(GP2) $\forall x \, (1 \cdot x = x \wedge x \cdot 1 = x)$

(GP3) $\forall x \, (x \cdot x^{-1} = 1 \wedge x^{-1} \cdot x = 1)$.

We could also axiomatize groups among all $\sigma_{sgrp} := (\cdot)$-structures: (GP1) stays the same while (GP2) and (GP3) are replaced with

(GP2*) $\exists v \forall x \, (v \cdot x = x \wedge x \cdot v = x)$

(GP3*) $\exists v \forall x \exists y \, (v \cdot x = x \wedge x \cdot v = x \wedge x \cdot y = v \wedge y \cdot x = v)$.

We can also axiomatize abelian groups, but the classes of cyclic groups (i.e. 1-generated) and non-cyclic groups are not axiomatizable. Indeed, cyclic groups are those in which $\exists x \forall y \, (\text{there is } n \in \mathbb{N}) \, (\underbrace{x \cdot x \cdot \ldots \cdot x}_{n} = y \vee \underbrace{x^{-1} \cdot x^{-1} \cdot \ldots \cdot x^{-1}}_{n} = y)$.

(e) Similarly in the signature $\sigma_{ring} := (0, 1, +, -(), \cdot)$ we define the theory RINGS axiomatizing the class of rings.

(f) In the same signature σ_{ring}, let FIELDS be the theory RINGS together with (FLD1) $\forall x\,(x \neq 0 \to \exists y\,(x \cdot y = 1 \wedge y \cdot x = 1))$
(FLD2) $\forall x \forall y\,(x \cdot y = y \cdot x)$

(g) For a prime number $p \in \mathbb{N}^+$, the theory of fields of characteristic p is $\text{FIELDS}_p := \text{FIELDS} \cup \{\underbrace{1+1+\ldots+1}_{p \text{ times}} = 0\}$.

Also, the fields of characteristic 0 are axiomatized by

$$\text{FIELDS}_0 := \text{FIELDS} \cup \{\underbrace{1+1+\ldots+1}_{p \text{ times}} \neq 0 \;:\; p \in \mathbb{N} \text{ is prime}\}.$$

(h) The theory ACF axiomatizes the class of algebraically closed fields: ACF := FIELDS $\cup$ the following infinitely many axioms: for each $n \in \mathbb{N}^+$,

$$\varphi_n := \forall u_0 \forall u_1 \ldots \forall u_n\,(u_n \neq 0 \to \exists x\,(u_0 + u_1 \cdot x + \ldots + u_n \cdot x^n = 0)),$$

where $x^k := \underbrace{x \cdot x \cdot \ldots \cdot x}_{k}$. A well-known model of ACF is the field of complex numbers, but there are countable models too, e.g. the algebraic closure of finite fields or of $\mathbb{Q}$.
We also denote by ACF_p, for p prime or 0, the theory $\text{ACF} \cup \text{FIELDS}_p$ of algebraically closed fields of characteristic p.

(i) Lastly, the theory of sets, called ZFC (= Zermelo–Fraenkel set theory with Choice), is an infinite theory in the signature $\sigma_{set} := (\in)$, where $\in$

is a binary rel. symbol, whose axioms state when two sets are equal, the existence of pairs, unions, definable subsets, powerset, an infinite set, and a couple more technical axioms, together with axiom of Choice. The list is a bit too long to give here, but can be found online, e.g. in my 20-page lecture notes "A quick intro to basic set theory" for undergraduates, available on our course webpage.

Important examples of theories come from concrete σ-structures:

Def. The **theory of a σ-structure** $\underline{A} := (A, \sigma)$ is the set

$$Th(\underline{A}) := \{\varphi \in \text{Sentences}(\sigma) : \underline{A} \models \varphi\}$$

of all σ-sentences true in $\underline{A}$.

Although theories of structures are really the main object of study in model theory, in the rest of mathematics, they are typically not useful since we can't usually tell which sentences are in $Th(\underline{A})$ for a structure $\underline{A}$.

Example. For the structure of arithmetic $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, we (humans) still don't know whether the sentences are in $Th(\underline{N})$ or not:

$$\text{GOLDBACH} := \forall x\, (\text{div}(2,x) \rightarrow \exists y\, \exists z\, (\text{prime}(y) \wedge \text{prime}(z) \wedge x = y + z)$$

and

$$\text{TWINPRIME} := \forall x\, \exists y\, (\underbrace{y \geq x}_{\substack{\exists z (x+z=y) \\ \parallel}} \wedge \text{prime}(y) \wedge \text{prime}(S(S(y)))).$$