# Ինտեգրալի հատկություններ

**Լեմմա** Դիցուք $f, g \in R[a,b]$: Այդ $\forall \varepsilon > 0$ $\exists f_\varepsilon \in C([a,b])$ այնպիսին, որ

$$\sup_{x\in[a,b]} \left| \int_a^x f(t)g(t)\,dt - \int_a^x f_\varepsilon(t) g(t)\,dt \right| < \varepsilon :$$

**Կիրառություն** $f, g \in R[a,b]$, $g$-ն մոնոտոն է $\Rightarrow \exists \xi \in [a,b]$

$$\int_a^b fg\,dx = g(a)\int_a^\xi f\,dx + g(b)\int_\xi^b f\,dx$$

$f \in C([a,b])$, $g \in C^1([a,b])$ $\Rightarrow$ ապացույցը հետևում է միջին արժեքի առաջին թեորեմից

$f \in R[a,b]$, $g \in C^1([a,b])$

Լեմմի համաձայն, $\exists \{f_n\} \subset C([a,b])$

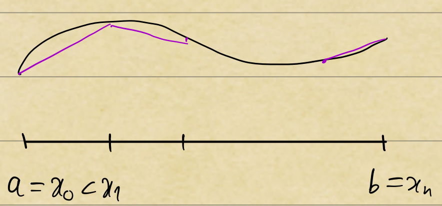
$$\sup_{x\in[a,b]} \left| \int_a^x f_n g\,dt - \int_a^x f g\,dt \right| \to 0$$

$\forall n \geq 1$ $\exists \xi_n \in [a,b]$ $\quad \int_a^b f_n g\,dx = g(a)\int_a^{\xi_n} f_n\,dx + g(b)\int_{\xi_n}^b f_n\,dx \quad (1)$

Կարող ենք ենթադրել, որ $\xi_n \to \xi$: Անցնենք սահմանի (1)-ում, երբ $n \to \infty$:

$$\int_a^{\xi_n} f_n\,dx \to \int_a^\xi f\,dx \; ; \qquad \underbrace{\int_a^{\xi_n} (f_n - f)\,dx}_{\to 0,\ n\to\infty} + \underbrace{\int_a^{\xi_n} f\,dx - \int_a^\xi f\,dx}_{\to 0,\ n\to\infty}$$

**Լեմմի ապացույց** $\quad g \equiv 1, \quad \left| \int_a^b f\,dx - \int_a^b f_\varepsilon\,dx \right| < \varepsilon$

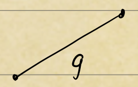



$x_k = a + \frac{b-a}{n}\cdot k, \quad k = 0, \dots, n, \quad P_n$

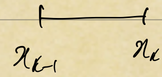
$f_n(x), \quad \xi_k = (x_k + x_{k-1})/2$

$f \in R[a,b] \Rightarrow \left| \int_a^b f\,dx - \sigma(f, P_n, \{\xi_k\}) \right| < \frac{\varepsilon}{3}$

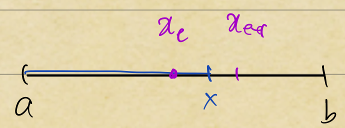
$$\left|\sigma(f, P_n, \{\xi_k\}) - \underbrace{\sigma(f_n, P_n, \{\xi_k\})}_{\substack{\| \\ \int_a^b f_n\,dx}}\right| = \left|\sum_{k=1}^{n}\left(f(\xi_k) - f_n(\xi_k)\right)\Delta x_k\right| \le \sum_{k=1}^{n}\omega(f, \Delta_k)\,\Delta x_k \to 0, \quad n\to\infty.$$



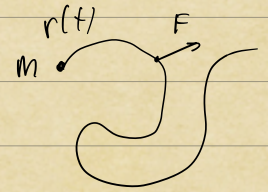

$$\int_{x_{k-1}}^{x_k} g\,dx = g\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x_k$$




$$\left|f(\xi_k) - f_n(\xi_k)\right| = \left|f(\xi_k) - \frac{f(x_{k-1}) + f(x_k)}{2}\right| \le \omega(f, \Delta_k) \qquad \blacksquare$$



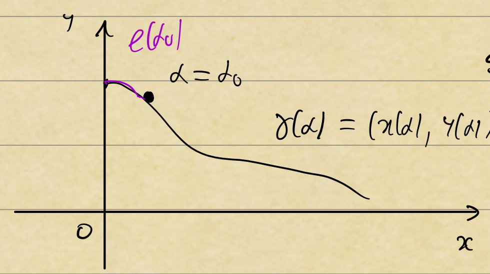

## Ինտեգրալի կիրառությունները




$$\forall [t_1, t_2] \subset \mathbb{R} \qquad \frac{m\,|\dot r(t_2)|^2}{2} - \frac{m\,|\dot r(t_1)|^2}{2} = \int_{t_1}^{t_2}\left(F(r(t)), \dot r(t)\right)dt$$

$$m\,\ddot r(t) = F(t)$$

Օրինակներ 1) Գլորվող գունդ




$s(t)$ գնդի գծային տեղափոխությունը սկզբնակետից $t$ պահին

$$\ell(\alpha_0) = \int_0^{\alpha_0} |\dot\gamma(\alpha)|\,d\alpha$$

$$s(t) = \ell(\alpha(t))$$

$$\boxed{\dot s(t)^2 + 2g\,h(s(t)) = C}, \qquad h(s) = y(\alpha(s))$$

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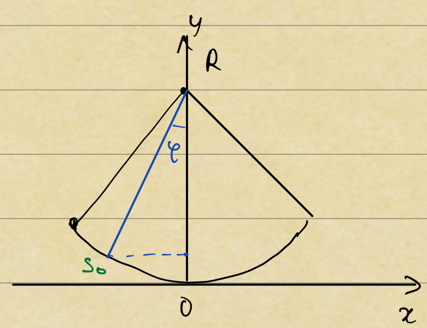



$$C = 2g\,h(s_0)$$

$$\varphi = \frac{s}{R} \Rightarrow h(s) = R - R\cos\varphi = R\left(1 - \cos\frac{s}{R}\right)$$

$$\dot s(t)^2 + 2gR\left(1 - \cos\frac{s}{R}\right) = 2gR\left(1 - \cos\frac{s_0}{R}\right)$$

$$\dot{s}(t)^2 = 2gR\left(\cos\frac{s}{R} - \cos\frac{s_0}{R}\right), \qquad s(t) = R\varphi(t), \qquad \varphi_0 = \frac{s_0}{R}$$

$$\dot{\varphi}(t)^2 = 2\frac{g}{R}(\cos\varphi - \cos\varphi_0); \qquad T$$ [illegible]

$$\varphi\left(\tfrac{T}{4}\right) = 0, \quad \dot{\varphi}(t) < 0, \quad 0 < t < \frac{T}{4}; \quad \cos\varphi = 1 - 2\sin^2\frac{\varphi}{2}$$

$$\dot{\varphi}(t) = -\sqrt{\frac{2g}{R}}\sqrt{2\sin^2\frac{\varphi_0}{2} - 2\sin^2\frac{\varphi}{2}}, \qquad 0 \le t \le \frac{T}{4}$$

$$\dot{\varphi}(t) = -2\sqrt{\frac{g}{R}}\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}\right)^{1/2}$$

$$-\frac{\dot{\varphi}(t)}{\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi(t)}{2}\right)^{1/2}} = 2\sqrt{\frac{g}{R}}, \qquad 0 < t \le \frac{T}{4}$$

$$-\int_0^{T/4}\frac{\dot{\varphi}(t)\,dt}{\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi(t)}{2}\right)^{1/2}} = 2\sqrt{\frac{g}{R}}\cdot\frac{T}{4}$$

$$\varphi(t) = r \qquad \int_0^{\varphi_0}\frac{dr}{\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{r}{2}\right)^{1/2}} = \frac{1}{2}\sqrt{\frac{g}{R}}\,T \Rightarrow T = 2\sqrt{\frac{R}{g}}\int_0^{\varphi_0}\frac{dr}{\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{r}{2}\right)^{1/2}}$$

$$T = 2\sqrt{\frac{R}{g}}\int_0^{\varphi_0}\frac{dr}{\sin\frac{\varphi_0}{2}\left(1 - \frac{\sin^2 r/2}{\sin^2\varphi_0/2}\right)^{1/2}} = 2\sqrt{\frac{R}{g}}\,2k\int_0^{\frac{\pi}{2}}\frac{\cos\theta\,d\theta}{k\cos\frac{r}{2}\left(1 - \sin^2\theta\right)^{1/2}}$$

$$\frac{\sin\frac{r}{2}}{\sin\frac{\varphi_0}{2}} = \sin\theta, \quad 0 < r < \varphi_0$$

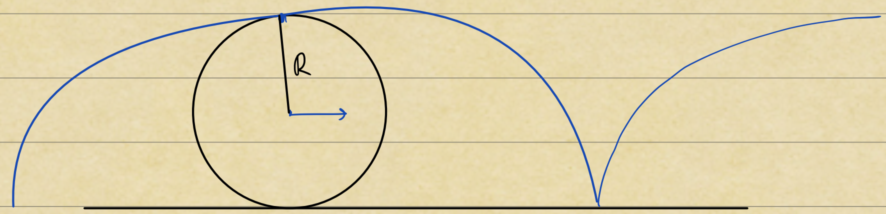
$$k = \sin\frac{\varphi_0}{2} \qquad \frac{1}{2k}\cos\frac{r}{2}\,dr = \cos\theta\,d\theta \Rightarrow dr = \frac{\cos\theta\,d\theta}{\cos\frac{r}{2}}\,2k$$

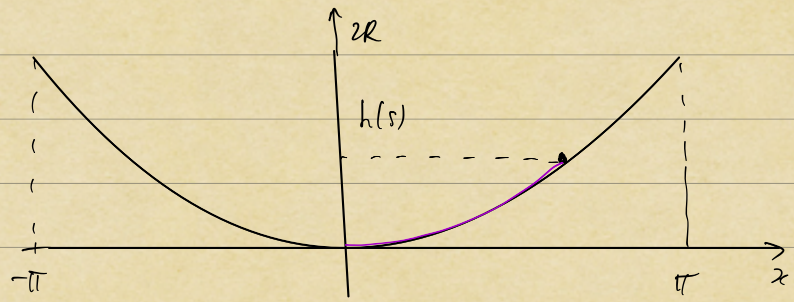
$$= 4\sqrt{\frac{R}{g}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1 - \sin^2\frac{r}{2}}} = 4\sqrt{\frac{R}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$$

$$\boxed{T = 4\sqrt{\frac{R}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \qquad k = \sin\frac{\varphi_0}{2}}$$

$$\varphi_0 \ll 1 \Rightarrow k = 0 \Rightarrow T = 2\pi\sqrt{\frac{R}{g}}$$

## 1) Ցիկլոիդ կամ բրախիստոխրոն




$$\alpha \in [-\pi, \pi]$$

$$\gamma(\alpha) = R\,(\alpha + \sin\alpha,\ 1 - \cos\alpha)$$

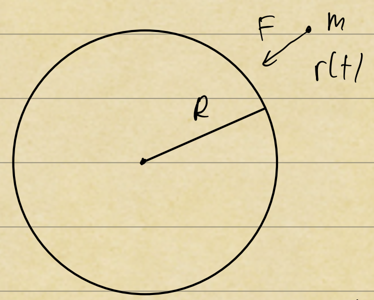
$$\ell(\alpha) = \int_0^{\alpha} R \sqrt{(1+\cos\beta)^2 + \sin^2\beta}\, d\beta$$

$$= R \int_0^{\alpha} \sqrt{\underbrace{2(1+\cos\beta)}_{2\cos^2\frac{\beta}{2}}}\, d\beta$$

$$= 2R \int_0^{\alpha} \cos\frac{\beta}{2}\, d\beta = 4R \sin\frac{\beta}{2}\Big]_0^{\alpha} = 4R\sin\frac{\alpha}{2}$$

Ապացուցել, որ $T = 4\pi\sqrt{\frac{R}{g}}$

## 2) Շարժում գրավիտացիոն դաշտում




$R = 6400$ կմ

$$m\,\ddot{r}(t) = F(r(t)) = -G\frac{Mm}{|r(t)|^3}\, r(t)$$

$$\frac{1}{2}m|\dot{r}(t)|^2 - \frac{1}{2}m|\dot{r}(t_0)|^2 = -GMm\int_{t_0}^{t} \frac{(r(s), \dot{r}(s))}{|r(s)|^3}\, ds$$

$\underline{n=2}$

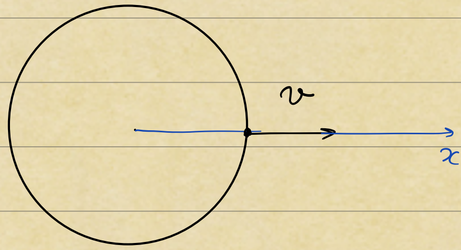
$$\frac{(r(s), \dot{r}(s))}{|r(s)|^3} = \frac{x(s)\dot{x}(s) + y(s)\dot{y}(s)}{(x(s)^2 + y(s)^2)^{3/2}} = \frac{\frac{1}{2}\frac{d}{ds}(x(s)^2 + y(s)^2)}{(x(s)^2 + y(s)^2)^{3/2}}$$

$$= -\frac{d}{ds}\left(x(s)^2 + y(s)^2\right)^{-\frac{1}{2}}$$

$$\frac{1}{2}|\dot{r}(t)|^2 - \frac{1}{2}|\dot{r}(t_0)|^2 = GM\int_{t_0}^{t} \frac{d}{ds}|r(s)|^{-1}\, ds = \frac{GM}{|r(t)|} - \frac{GM}{|r(t_0)|}$$

$$\frac{GM}{R^2} = g \Rightarrow GM = gR^2$$

$$\boxed{\frac{1}{2}|\dot{r}(t)|^2 - \frac{gR^2}{|r(t)|} = C}$$




$r(t) = (x(t), 0, 0)$ , $x(0) = R$, $\dot{x}(0) = v$

$C = \frac{1}{2}v^2 - gR$

$$\frac{1}{2}\dot{x}(t)^2 = C + \frac{gR^2}{x(t)}$$

$$\dot{x}(t)^2 = v^2 - 2gR + \frac{2gR^2}{x(t)}$$

$v = \sqrt{2gR} \Rightarrow \begin{cases} \dot{x}(t) = \sqrt{2g}\,R\,\frac{1}{\sqrt{x(t)}} \\ x(0) = R \end{cases} \Rightarrow x(t) \to +\infty$, per $t \to +\infty$

$v = 11{,}2$ km/s