Ինտեգրալ և ածանցյալ

- **Նյուտոն-Լայբնիցի բանաձևը**

$$F \in C^1([a,b]) \Rightarrow \int_a^b F'(x)\,dx = F(b) - F(a)$$

- **Մասերով ինտեգրում**

$$u, v \in C^1([a,b]) \Rightarrow \int_a^b u'v\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u v'\,dx$$

- **Փոփոխականի փոխարինում**

$f \in R([a,b])$, $\varphi \in C^1([\alpha,\beta])$, $\varphi([\alpha,\beta]) \subset [a,b]$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$,

$\forall\, t_1, t_2 \in [\alpha,\beta] \quad t_1 < t_2 \Rightarrow \varphi(t_1) < \varphi(t_2)$

Ապա

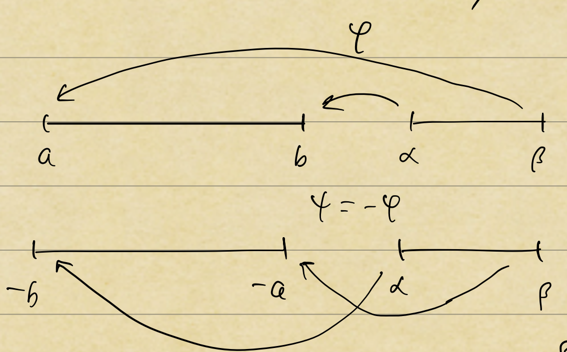
$$\int_a^b f(x)\,dx \underset{\substack{x = \varphi(t) \\ dx = \varphi'(t)\,dt}}{=} \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Դիտարկում $\quad \forall\, t_1, t_2 \in [\alpha,\beta] \quad \varphi(t_1) > \varphi(t_2)$, $\varphi(\alpha) = b$, $\varphi(\beta) = a$

$$\int_a^b f(x)\,dx = \int_\beta^\alpha f(\varphi(t))\,\varphi'(t)\,dt = -\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$g(y) = f(-y)$, $y \in [-b,-a]$; $\psi(t) = -\varphi(t)$, $\psi : [\alpha,\beta] \to \mathbb{R}$

ψ աճող է, $\psi(\alpha) = -b$, $\psi(\beta) = -a$



$$\int_{-b}^{-a} g(y)\,dy = \int_\alpha^\beta g(\psi(t))\,\psi'(t)\,dt$$

$$\int_a^b f(x)\,dx = -\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Ապացույց

$$\int_a^b f(x)\,dx = \int_\alpha^\beta \underbrace{f(\varphi(t))\,\varphi'(t)}_{g(t)}\,dt$$

$$\tau_1 \quad \tau_2 \quad \tau_n \qquad\qquad \xi_1 = \varphi(\tau_1) \quad \varphi(\tau_2) = \xi_2 \quad \varphi(\tau_n) = \xi_n$$

$$P' \quad \alpha = t_0 \quad t_1 \quad t_2 \quad t_{n-1} \quad t_n = \beta \qquad a = x_0 < x_1 < \dots \quad x_{n-1} < b = x_n \quad P$$

$$x_k = \varphi(t_k), \quad k = 0, \dots, n$$

$$\sigma(g, P', \tau) = \sum_{k=1}^{n} g(\tau_k)\, \Delta \tau_k = \sum_{k=1}^{n} f(\varphi(\tau_k))\, \varphi'(\tau_k)(t_k - t_{k-1})$$

$$\sigma(f, P, \underset{\varphi(\tau)}{\underset{\shortparallel}{\xi}}) = \sum_{k=1}^{n} f(\varphi(\tau_k))\, \Delta x_k = \sum_{k=1}^{n} f(\varphi(\tau_k))(\varphi(t_k) - \varphi(t_{k-1}))$$

$$\lambda(P') \to 0 \iff \lambda(P) \to 0$$

$$\forall d > 0 \quad \exists d' > 0 \qquad \lambda(P') < d' \Rightarrow \lambda(P) < d$$

$$\forall d' > 0 \quad \exists d > 0 \qquad \lambda(P) < d \Rightarrow \lambda(P') < d'$$

$$a \qquad b$$

$$\uparrow$$

$$\alpha \qquad \beta$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall t_1, t_2 \in [\alpha, \beta] \quad |t_1 - t_2| < \delta \Rightarrow |\varphi(t_1) - \varphi(t_2)| < \varepsilon$$

$$d > 0 \to d' > 0 \qquad \forall s_1, s_2 \in [\alpha, \beta] \quad |s_1 - s_2| < d' \Rightarrow |\varphi(s_1) - \varphi(s_2)| < d$$

$$\lambda(P') < d' \iff \forall k \in [\![1, n]\!] \quad |t_k - t_{k-1}| < d'$$

$$\Rightarrow \forall k \in [\![1, n]\!] \quad |x_k - x_{k-1}| < d \Rightarrow \lambda(P) < d$$

$$\text{Лагранж} \Rightarrow \forall k \in [\![1, n]\!] \quad \exists \tau_k' \in \underbrace{[t_{k-1}, t_k]}_{\Delta_k} \qquad \varphi'(\tau_k')(t_k - t_{k-1}) = x_k - x_{k-1}$$

$$\sigma(g, P', \tau) - \sigma(f, P, \underset{\varphi(\tau)}{\underset{\shortparallel}{\xi}}) = \sum_{k=1}^{n} f(\varphi(\tau_k))(\varphi'(\tau_k) - \varphi'(\tau_k'))\, \Delta t_k$$

$$+ \sum_{k=1}^{n} f(\varphi(\tau_k))\, \varphi'(\tau_k')\, \Delta t_k$$

$$- \sum_{k=1}^{n} f(\varphi(\tau_k))(\varphi(t_k) - \varphi(t_{k-1}))$$

$$|\sigma(g, P', \tau) - \sigma(f, P, \xi)| \le \underset{\sup\limits_{[a,b]} |f|}{\underset{\shortparallel}{M}} \sum_{k=1}^{n} \omega(\varphi', \Delta_k)\, \Delta t_k$$

$$\lambda(P') \to 0 \Rightarrow \sum_{k=1}^{n} \omega(\varphi', \Delta_k)\, \Delta t_k \to 0, \quad \sigma(g, P', \tau) \to \int_\alpha^\beta g(t)\,dt \quad \sigma(f, P, \xi) \to \int_a^b f\,dx$$

$$\alpha \mapsto a, \quad \beta \mapsto b, \quad \varphi \in C^1([\alpha, \beta]), \quad \varphi([\alpha, \beta]) \subset [a, b]$$

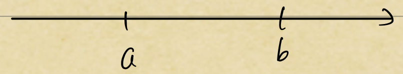
$t \in [0,1]$, $\varphi(t) = t^3 \left(\sin \frac{1}{t}\right)^2$

Կիրառություններ 1) $f, g \in R[a,b]$, $g \geq 0$ կամ $g \leq 0$

$$\int_a^b f g \, dx = \underset{\substack{\vee \\ f(\xi)}}{\mu} \int_a^b g \, dx, \quad \mu \in \left[\inf_{[a,b]} f, \sup_{[a,b]} f\right]$$

$g \equiv 1$, $f(t)$ շարժվող կետի ակնթարթային արագությունն է t պահին

$$\int_a^b f(t)\, dt = f(\tau)(b-a)$$



2)

$f \in C([a,b])$, $F(x) = \int_a^x f(t)\, dt$, $\forall x \in [a,b]$ $F'(x) = f(x)$

$$G(x) = \int_x^b f(t)\, dt \underset{\substack{s = a+b-t \\ t = a+b-s}}{=} -\int_{a+b-x}^{a} f(a+b-s)\, ds = \int_a^{a+b-x} f(a+b-s)\, ds$$

$$\frac{d}{dx} \int_a^{h(x)} f(t)\, dt = \frac{d}{dx} F(h(x)) = F'(h(x))\, h'(x) = f(h(x))\, h'(x)$$

$$G'(x) = f(a+b-(a+b-x)) \times (-1) = -f(x)$$

$$\boxed{\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\, dt = f(h(x))\, h'(x) - f(\varphi(x))\, \varphi'(x)}$$

3) $f \in C(\mathbb{R})$, $[a,b] \subset \mathbb{R}$, $\varepsilon > 0 \rightarrow f_\varepsilon(x)$, $f_\varepsilon \in C^1([a,b])$,

$$\sup_{x \in [a,b]} |f_\varepsilon(x) - f(x)| < \varepsilon$$

$$F_\delta(x) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t)\, dt$$

- $\forall \delta > 0$ $F_\delta \in C^1(\mathbb{R})$

- $\forall [a,b]$ $\sup_{x \in [a,b]} |F_\delta(x) - f(x)| \to 0$, $\delta \to 0$

$$F_\delta(x)-f(x)=\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\,dt-f(x)=\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\left(f(t)-f(x)\right)dt$$

$\forall \varepsilon>0 \quad \exists \delta>0 \quad \forall h\in\mathbb{R} \qquad |h|<\delta \Rightarrow |f(x+h)-f(x)|<\varepsilon$

$\mathbb{R}$ $\quad \forall R>0 \qquad \forall x\in[-R,R]$

$$|F_\delta(x)-f(x)|\le\frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\underbrace{|f(t)-f(x)|}_{<\varepsilon}dt<\varepsilon \Rightarrow \sup_{|x|\le R}|F_\delta(x)-f(x)|\le\varepsilon.$$

$|x|\le R,\ [a,b]\subset[-R,R]$

Թեյլորի բանաձևը

Թեորեմ Դիցուք $f\in C^n((a,b))$, $n\ge 1$: Այդ $\forall\, c,x\in(a,b)$

$$\boxed{f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k+r_{n-1}(x,c),\quad r_{n-1}(x,c)=\frac{1}{(n-1)!}\int_c^x f^{(n)}(t)(x-t)^{n-1}dt}$$

$a \quad c \quad x \quad b$

$$r_{n-1}(x,c)=\frac{1}{(n-1)!}\underset{\xi_{c,x}\in(c,x)}{f^{(n)}(\xi)}\int_c^x\underbrace{(x-t)^{n-1}}_{\frac{d}{dt}\left(-\frac{(x-t)^n}{n}\right)}dt$$

$$=\frac{1}{n!}f^{(n)}(\xi_{c,x})(x-c)^n$$

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{1}{n!}f^{(n)}(\xi_{c,x})(x-c)^n$$

$$f^{(n)}(\xi_{c,x}) = f^{(n)}(c)+\underbrace{\left(f^{(n)}(\xi_{c,x})-f^{(n)}(c)\right)}_{o(1),\ \text{երբ}\ x\to c}$$

$$\boxed{f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k+o(|x-c|^n)}$$

$n=1 \qquad f(x)=f(c)+\int_c^x f'(t)\,dt$

Թեորեմի ապացույց

$$f(x)-f(c)=\int_c^x f'(t)\,dt = -\int_c^x f'(t)(x-t)'\,dt$$

$$= -f'(t)(x-t)\Big|_{t=c}^{t=x} + \int_c^x f''(t)(x-t)\,dt$$

$$= f'(c)(x-c) + \underbrace{\int_c^x f''(t)(x-t)\,dt}_{r_1(x,c)}$$

$$= f'(c)(x-c) - \int_c^x f''(t)\frac{d}{dt}\frac{(x-t)^2}{2}\,dt$$

$$= f'(c)(x-c) - f''(t)\frac{(x-t)^2}{2}\Big|_{t=c}^{t=x} + \frac{1}{2}\int_c^x f'''(t)(x-t)^2\,dt$$

$$= f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + r_2(x,c)$$

$$r_{n-1}(x,c) = \frac{1}{(n-1)!}\int_c^x f^{(n)}(t)(x-t)^{n-1}\,dt$$

$$= \frac{1}{(n-1)!}\int_c^x f^{(n)}(t)\,\frac{d}{dt}\left(-\frac{(x-t)^n}{n}\right)dt$$

$$= -\frac{1}{n!}f^{(n)}(t)(x-t)^n\Big|_{t=c}^{t=x} + \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt$$

$$= \frac{1}{n!}f^{(n)}(c)(x-c)^n + r_n(x,c)$$