Math Logic: Model Theory & Computability
Lecture 03

Def. For a σ-structure $\underline{B}$, we say that a subset $A \subseteq B$ *supports a substructure* if there is a substructure of $\underline{B}$ with underlying set A. We abuse terminology and just say that A *is a substructure*.

Obs. Let $\underline{B}$ be a σ-structure and $A \subseteq B$.

(a) A can only support at most one substructure.

(b) A supports a substructure $\Leftrightarrow$ A contains the constants of $\underline{B}$ and is closed under the functions of $\underline{B}$, i.e.
- $c^{\underline{B}} \in A$ for each $c \in \mathrm{Const}(\sigma)$,
- $f^{\underline{B}}(A^n) \subseteq A$ for each n-ary $f \in \mathrm{Funct}(\sigma)$.

(c) In particular, if σ has no constant and function symbols, then every subset supports a substructure.

Prop. Arbitrary (maybe unctbl) intersections of substructures is a substructure.

Proof. If $\underline{B}$ is a σ-structure and $\{\underline{A}_i : i \in I\}$ is a family of substructures of $\underline{B}$ with underlying sets A_i, then $\bigcap_{i \in I} A_i$ is a substructure by (b) of Obs. above because each A_i contains the constants of $\underline{B}$ and each A_i is closed under the functions of $\underline{B}$. $\square$

Let $\underline{B}$ be a σ-structure and $S \subseteq B$. Since there is always at least one substr. of $\underline{B}$ containing S, namely $\underline{B}$ itself, we can define the *substructure generated by S* as the intersection of all substructures contain S, thus the $\subseteq$-smallest substructure containing S. We denote it by $\langle S \rangle_{\underline{B}}$.

This is a top-down definition, which is hard to work with, so we give a more constructive bottom-up equivalent:

Prop. Let $\mathcal{B}$ be a σ-structure and $S \subseteq B$.

(a) $\langle S \rangle_{\mathcal{B}} = \bigcup\limits_{n \in \mathbb{N}}^{\uparrow} S_n$, where $S_0 := S \cup \{c^{\mathcal{B}} : c \in \mathrm{Const}(\mathcal{B})\}$ and

$$S_{n+1} := S_n \cup \bigcup_{f \in \mathrm{Funct}(\sigma)} f^{\mathcal{B}}(S_n^{a(f)}).$$

(b) $|\langle S \rangle_{\mathcal{B}}| \leq \max(|\mathbb{N}|, |S|, |\mathrm{Const}(\sigma)| + |\mathrm{Funct}(\sigma)|) \leq \max(\aleph_0, |S|, |\sigma|)$, where $\aleph_0 := |\mathbb{N}|$, $|\sigma| := |\mathrm{Const}(\sigma)| + |\mathrm{Funct}(\sigma)| + |\mathrm{Rel}(\sigma)|$.

Proof. (a) It is clear by induction on n that $S_n \subseteq \langle S \rangle_{\mathcal{B}}$ for all $n \in \mathbb{N}$, so $\bigcup\limits_{n \in \mathbb{N}}^{\uparrow} S_n \subseteq \langle S \rangle_{\mathcal{B}}$. To show that $\langle S \rangle_{\mathcal{B}} \subseteq \bigcup\limits_{n \in \mathbb{N}}^{\uparrow} S_n$ we just need to show that $\bigcup\limits_{n \in \mathbb{N}}^{\uparrow} S_n$ is a substructure.

It contains constants of $\mathcal{B}$ because S_0 does. As for closedness under the functions of $\mathcal{B}$, fix a k-ary $f \in \mathrm{Funct}(\sigma)$ and take elements $x_1, x_2, \dots, x_k \in \bigcup\limits_{n \in \mathbb{N}}^{\uparrow} S_n$. But then each $x_i \in S_{n_i}$, so all $x_1, x_2, \dots, x_k \in S_n$ where $n := \max\{n_1, n_2, \dots, n_k\}$. Hence $f^{\mathcal{B}}(x_1, x_2, \dots, x_k) \in S_{n+1} \subseteq \bigcup\limits_{i \in \mathbb{N}}^{\uparrow} S_i$. $\boxed{a}$

(b) It is enough to prove by induction on n that $|S_n| \leq \max(\aleph_0, |S|, |\sigma|)$ because then $|\bigcup\limits_{n \in \mathbb{N}}^{\uparrow} S_n| \leq \aleph_0 \cdot \max(\aleph_0, |S|, |\sigma|) \leq \max(\aleph_0, |S|, |\sigma|)$.

In this last statement and the induction above what we use is the following statement from set theory:

Cardinality of products: If A, B are sets one of which is infinite, then $|A \times B| \leq \max(|A|, |B|)$. $\boxed{b}$ $\square$

Example. In $\underline{R} := (\mathbb{R}, 0, 1, +, \cdot)$ with standard interpretations,
$\langle \emptyset \rangle_{\underline{R}} = (\mathbb{N}, 0, 1, +, \cdot)$, $\langle \{-1\} \rangle_{\underline{R}} = (\mathbb{Z}, 0, 1, +, \cdot)$.

Reducts and expansions.

There is also another kind of sub-object of structure we can define:

Def. Let $\sigma_0 \subseteq \sigma_1$ be signatures and $\underline{A}$ be a σ_0-structure, $\underline{B}$ be a σ_1-structure. We say that $\underline{A}$ is the reduct of $\underline{B}$ to a σ_0-structure or that $\underline{B}$ is an expansion of $\underline{A}$ to a σ_1-structure if $A = B$ and for each symbol $s \in \sigma_0$, $s^{\underline{A}} = s^{\underline{B}}$.
We denote the reduct of $\underline{B}$ to a σ_0-structure by $\underline{B}|_{\sigma_0}$, noting that it is unique.

Example. $(\mathbb{R}, 0, +)$ is the reduct of $(\mathbb{R}, 0, 1, +, \cdot)$, which is the reduct of $(\mathbb{R}, 0, 1, +, \cdot, <)$, with the standard interpretations.

Homomorphisms.

Vector notation. For a set A and $n \in \mathbb{N}$, we denote the elements of A^n by $\vec{a} := (a_1, a_2, \dots, a_n)$, where we denote $(\vec{a})_i := a_i$. We denote by $|\vec{a}|$ the length of $\vec{a}$, i.e. n. If $h: A \to B$, and $\vec{a} \in A^n$, we write $h(\vec{a})$ to mean $(h(a_1), h(a_2), \dots, h(a_n)) \in B^n$.

Def. For σ-structures $\underline{A} := (A, \sigma)$, $\underline{B} := (B, \sigma)$, a function $h: A \to B$ is called a σ-homomorphism, and is written $h: \underline{A} \to \underline{B}$, if

(i) $h(c^{\underline{A}}) = c^{\underline{B}}$ for each $c \in \mathrm{Const}(\sigma)$

(ii) $h \circ f^{\underline{A}} = f^{\underline{B}} \circ h$, i.e. for each k-ary $f \in \mathrm{Funct}(\sigma)$ and $\vec{a} \in A^k$,

$$h\left(f^{\underline{A}}(\vec{a})\right) = f^{\underline{B}}\left(h(\vec{a})\right).$$

(iii) for each k-ary $R \in \mathrm{Rel}(\sigma)$ and $\vec{a} \in A^k$,

$$\vec{a} \in R^{\underline{A}} \implies h(\vec{a}) \in R^{\underline{B}}.$$

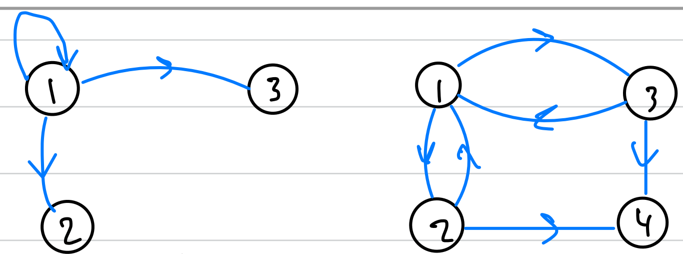
Examples. (a) $\underline{\mathbb{R}} := (\mathbb{R}, 1, \cdot, ()^{-1})$ be the group of reals under $+$ in the signature $\sigma_{gp} := (1, \cdot, ()^{-1})$ where $1^{\underline{\mathbb{R}}} := 0$, $\cdot^{\underline{\mathbb{R}}} := +$, $(()^{-1})^{\underline{\mathbb{R}}} := -()$. Also let $\underline{\mathbb{R}}^+ := (\mathbb{R}^+, 1, \cdot, ()^{-1})$, where $\mathbb{R}^+ := (0, \infty)$, $1^{\underline{\mathbb{R}}^+} := 1$, $\cdot^{\underline{\mathbb{R}}^+} := \cdot$, $(()^{-1})^{\underline{\mathbb{R}}^+} := ()^{-1}$. Then

$$h : \mathbb{R} \to \mathbb{R}^+, \quad x \mapsto 2^x$$

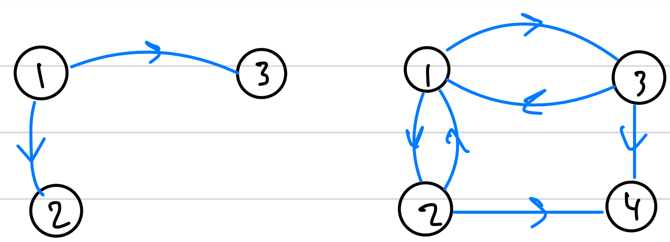
is a σ_{gp}-homomorphism.

This h is a bijection (hence an isomorphism, defined below).

(b) In the signature $\sigma_{gph} := (E)$, let $\iota : A \hookrightarrow B$ be the inclusion map for sets $A \subseteq B$, where $A = \{1, 2, 3\}$, $B := \{1, 2, 3, 4\}$. Then



ι is not a hom.



ι is a hom.