Math Logic: Model Theory & Computability
Lecture 02

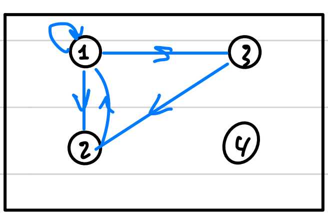
Recall. An *n-ary operation* on a set S is just a function $S^n \to S$.
Similarly, an *n-ary relation* on a set S is just a subset of S^n.
Also, for $n \in \mathbb{N} := \{0,1,2,\dots\}$, $S^n := \underbrace{S \times S \times \dots \times S}_{n}$ if $n \geq 1$ and $S^0 := \{\emptyset\}$.

For a signature $\sigma := (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$, let $\mathrm{Const}(\sigma) := \mathcal{C}$, $\mathrm{Func}(\sigma) := \mathcal{F}$, $\mathrm{Rel}(\sigma) := \mathcal{R}$, $a_\sigma := a$.

Def. Let σ be a signature. A *σ-structure* is a pair $\underline{S} := (S, i)$, where S is a set, called the *underlying set* or *universe* of $\underline{S}$, and i is "a map assigning to the symbols in σ their interpretation in $\underline{S}$", more precisely,
- for each $c \in \mathrm{Const}(\sigma)$, $i(c) \in S$.
- for each $f \in \mathrm{Func}(\sigma)$, $i(f)$ is an $a_\sigma(f)$-ary operation on S.
- for each $R \in \mathrm{Rel}(\sigma)$, $i(R)$ is an $a_\sigma(R)$-ary relation on S.

We call this i an *interpretation* of σ in $\underline{S}$.

Examples. (a) A signature for graphs is $\sigma_{gph} := (\emptyset, \emptyset, \{E\}, (E \mapsto 2))$, i.e. it contains just one binary relation symbol. This notation is too annoying, so we will write $\sigma_{gph} := (\dot{E})$ instead and then say "where E is a binary relation symbol".
Now let's give an example of a σ_{gph}-structure, i.e. a directed graph.



$\underline{G}$

Let $\underline{G} := (V, i)$ where $V := \{1,2,3,4\}$ and the interpretation $i(E) := \{(1,1), (1,2), (2,1), (1,3), (3,2)\}$.
This is again unintuitive notation, so we write $\underline{G} := (V, E^{\underline{G}})$ where $E^{\underline{G}}$ is the interpretation of E defined by $E^{\underline{G}} := \{(1,1), (1,2), (2,1), (1,3), (3,2)\}$.

(b) The signature for groups is $\sigma_{gp} := (\{1\}, \{\cdot, ()^{-1}\}, \emptyset, (\cdot \mapsto 2, ()^{-1} \mapsto 1))$.
Instead we write $\sigma_{gp} := (1, \cdot, ()^{-1})$, where 1 is a const. symb., $\cdot$ is a binary function symb., and $()^{-1}$ is a unary (i.e. arity $= 1$) func. symbol.
Now we give examples of σ_{gp}-structures:

- $\underline{Z} := (\mathbb{Z}, 1^{\underline{Z}}, \cdot^{\underline{Z}}, ()^{-1\underline{Z}})$, where $1^{\underline{Z}} := 0$, $\cdot^{\underline{Z}} :=$ addition, $()^{-1\underline{Z}} := -() := (x \mapsto -x)$. This σ_{gp}-structure is actually a group.

- $\underline{Z} := (\mathbb{Z}, 1^{\underline{Z}}, \cdot^{\underline{Z}}, ()^{-1\underline{Z}})$, where $1^{\underline{Z}} := 7$, $\cdot^{\underline{Z}} := ((x,y) \mapsto (|x|+1)^{|y|} - 8)$, $()^{-1\underline{Z}} := (x \mapsto 1000)$. This is a σ_{gp}-structure, but it is very much NOT a group.

- $\underline{M} := (GL_n(\mathbb{F}), 1^{\underline{M}}, \cdot^{\underline{M}}, ()^{-1\underline{M}})$, where $1^{\underline{M}} := \left.\overbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}^{n}\right\} n$, $\cdot^{\underline{M}} :=$ matrix multiplication, $()^{-1\underline{M}} :=$ matrix inversion.
This σ_{gp}-structure is a group.

(c) Let $\sigma_{ring} := (0, 1, +, -(), \cdot)$ be the signature for rings/fields, where $0, 1$ are constant symbols, $+, \cdot$ are binary funct. symbols, $-()$ is a unary funct. symbol.
Here are some examples of σ_{ring}-structures:

- $\underline{Z} := (\mathbb{Z}, 0^{\underline{Z}}, 1^{\underline{Z}}, +^{\underline{Z}}, -()^{\underline{Z}}, \cdot^{\underline{Z}})$ with the standard interpretation of symbols. This a σ_{ring}-structure and it is also a ring.

- $\underline{R} := (\mathbb{R}, 0^{\underline{R}}, 1^{\underline{R}}, +^{\underline{R}}, -()^{\underline{R}}, \cdot^{\underline{R}})$, where $0^{\underline{R}} := \sqrt{2}$, $1^{\underline{R}} := 0$, $+^{\underline{R}}(x,y) := \sin x - \cos y$, $-()^{\underline{R}} := (x \mapsto |x|)$, $\cdot^{\underline{R}}(x,y) := x - y$.
This is a σ_{ring}-structure but it's not a ring.

- $\underline{R} := (\mathbb{R}, 0^{\underline{R}}, 1^{\underline{R}}, +^{\underline{R}}, -()^{\underline{R}}, \cdot^{\underline{R}})$ with standard interpretations is a σ_{ring}-structure that is a field.

Notation abuse. It's too long and ugly to write $\underline{M} := (GL_n(\mathbb{F}), 1^{\underline{M}}, \cdot^{\underline{M}}, ()^{-1^{\underline{M}}})$, so instead we write $\underline{M} := (GL_n(\mathbb{F}), 1, \cdot, ()^{-1})$ and then specify how each symbol is interpreted, i.e. define $1^{\underline{M}}, \cdot^{\underline{M}}, ()^{-1\underline{M}}$.

Substructures.

Recall. For a function $f: X \to Y$ and a subset $X_0 \subseteq X$, we define the restriction of f to X_0 as the function $f|_{X_0}: X_0 \to Y$ given by $x \mapsto f(x)$.

Def. Let $\underline{A} := (A, \sigma)$ and $\underline{B} := (B, \sigma)$ be two σ-structures, for a signature σ. We say that $\underline{A}$ is a substructure of $\underline{B}$, and write $\underline{A} \subseteq \underline{B}$, if

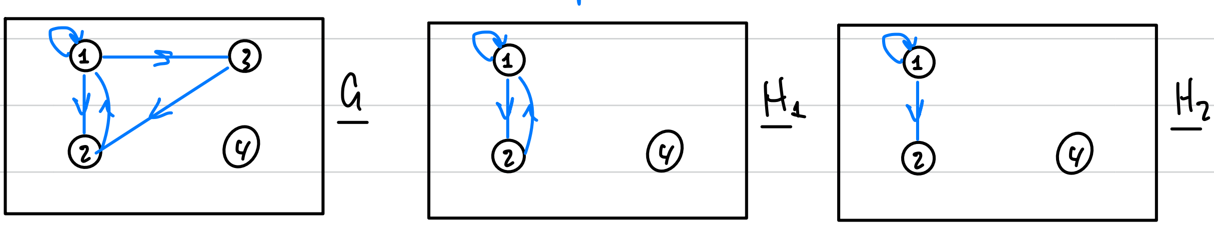
(i) $A \subseteq B$.

(ii) For each $c \in Const(\sigma)$, $c^{\underline{A}} = c^{\underline{B}}$.

(iii) For each $f \in Func(\sigma)$ of arity n, $f^{\underline{A}} = f^{\underline{B}}|_{A^n}$. (This in particular implies that $f^{\underline{B}}(A^n) \subseteq A$.)

(iv) For each $R \in Rel(\sigma)$ of arity n, $R^{\underline{A}} = R^{\underline{B}} \cap A^n$.

Examples. (a) For a graph $\underline{G} := (V, E^{\underline{G}})$, not all subgraphs (in the sense of graph theory) are σ_{gph}-substructures. Indeed, for $\underline{H} := (U, E^{\underline{H}})$ with $U \subseteq V$, $\underline{H}$ is a substructure of $\underline{G}$ if $E^{\underline{H}} = E^{\underline{G}} \cap U^2$, i.e. exactly when $\underline{H}$ is the induced subgraph on U.



$\underline{H_1} \subseteq \underline{G}$ $\quad$ $\underline{H_2} \not\subseteq \underline{G}$ not a substructure

(b) For a τ_{gp}-structure $\underline{\Gamma} := (\Gamma, 1, \cdot, (\,)^{-1})$, if $\underline{\Gamma}$ is a group then its substructures are precisely its subgroups.

(c) Let $\tau_{sgp} := (\cdot)$ be signature for semi-groups and let $\underline{Z} := (\mathbb{Z}, \cdot^{\underline{Z}})$, where $\cdot^{\underline{Z}}$ is defined as the usual addition on $\mathbb{Z}$. The following are substructures of $\underline{Z}$:

- $(\mathbb{N}, \cdot^{\underline{Z}})$, i.e. $\mathbb{N}$ with usual addition.
- $(\{7, 8, 9, \dots\}, \cdot^{\underline{Z}})$
- $(\{0\}, \cdot^{\underline{Z}})$
- $(m\mathbb{N}, \cdot^{\underline{Z}})$ where $m \in \mathbb{Z}$.
- $(\{\dots, -7, -6, -5\}, \cdot^{\underline{Z}})$
- $(d\mathbb{N} + k \cdot d, \cdot^{\underline{Z}})$, where $d \in \mathbb{Z}$, $k \geq 0$.
- $(\{0\} \cup (d\mathbb{N} + k \cdot d), \cdot^{\underline{Z}})$, where $d \in \mathbb{Z}$, $k \geq 0$
- $(d\mathbb{Z}, \cdot^{\underline{Z}})$, $d \in \mathbb{Z}$.

Complete classification?
No $\{2\} \cup (\mathbb{N} + 4)$.