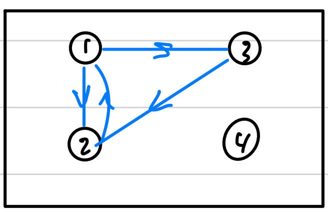
Math Logic: Model Theory & Computability
Lecture 01

Model Theory

Mathematical structures.

Every mathematician recognizes a mathematical structure when he/she sees one.

Examples. (a) A graph is a pair $G := (V, E)$, where V is a set (of vertices) and $E \subseteq V^2$ is a binary relation on V, called the set of edges.



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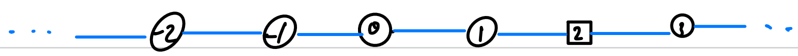

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(b) A partial order is a pair $(X, \leq)$, where X is a set and $\leq$ is a binary relation on X (i.e. a subset of X^2) satisfying certain axioms:

(i) Reflexivity: $x \leq x$ for all $x \in X$.

(ii) Anti-symmetry: $x \leq y$ and $y \leq x$ then $x = y$ for all $x, y \in X$.

(iii) Transitivity: $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in X$.

For example the usual $\leq$ on $\mathbb{Z}$, or the subset relation $\subseteq$ on $\mathcal{P}(\mathbb{N}) :=$ the powerset of a set $\mathbb{N}$.

(c) A group is a quadruple $\Gamma := (\Gamma, 1, \cdot, ()^{-1})$, where Γ is a set, 1 is an element of Γ, $\cdot$ is a binary operation on Γ, i.e. $\cdot : \Gamma^2 \to \Gamma$, and $()^{-1}$ is a unary operation on Γ, i.e. $()^{-1} : \Gamma \to \Gamma$, satisfying the following conditions:

(i) $\cdot$ is associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \Gamma$.
(ii) 1 is an identity for $\cdot$: $1 \cdot x = x \cdot 1 = x$ for all $x \in \Gamma$.
(iii) $(\ )^{-1}$ is an inverse for $\cdot$: $x^{-1} \cdot x = x \cdot x^{-1} = 1$ for all $x \in \Gamma$.

For example, $\underline{\mathbb{Z}} := (\mathbb{Z}, 0, +, -(\ ))$, $\mathrm{Sym}(n)$ is a non-abelian group of all permutations of $\{0, 1, \dots, n-1\}$ with the operation of composition, $GL_n(\mathbb{R}) :=$ the group of invertible $n \times n$ matrices over $\mathbb{R}$ under multiplication.

(d) A ring is a 6-tuple $\underline{R} := (R, 0, 1, +, -(\ ), \cdot)$ where $(R, 0, +, -(\ ))$ is an abelian group, $1 \in R$, $\cdot$ is a binary operation satisfying:
(i) $\cdot$ is associative
(ii) 1 is an identity for $\cdot$, i.e. $1 \cdot x = x \cdot 1 = x$ for all $x \in R$.
(iii) $\cdot$ distributes over $+$, i.e. $x \cdot (y+z) = xy + xz$ and $(y+z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in R$.
(iv) $0 \neq 1$.

For example, the ring of integers $\underline{\mathbb{Z}} := (\mathbb{Z}, 0, 1, +, -(\ ), \cdot)$, the ring of $n \times n$ matrices $M_n(\mathbb{R})$ over $\mathbb{R}$, the ring of polynomials $\mathbb{F}[t]$ over a field $\mathbb{F}$, the ring of continuous functions $f : [0,1] \to \mathbb{R}$ under usual addition and multiplication, the ring of linear transformations $V \to V$ for a vector space V over a field $\mathbb{F}$ under the operations of addition (as $+$) and composition (as $\cdot$).

(e) A field is a ring $\underline{F} := (\mathbb{F}, 0, 1, +, -(\ ), \cdot)$ such that
(i) every nonzero element has a $\cdot$-inverse, i.e. for all nonzero $x \in \mathbb{F}$ there is $y \in \mathbb{F}$ such that $x \cdot y = y \cdot x = 1$.
(ii) $\cdot$ is commutative: $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{F}$.

(f) An ordered field is a 7-tuple $\underline{R} := (R, 0, 1, +, -(), \cdot, \leq)$, where $(R, 0, 1, +, -(), \cdot)$ is a field, $(R, \leq)$ is a total/linear order (i.e. a partial order s.t. for all $x, y \in R$, $x \leq y$ or $y \leq x$) satisfying:

(i) $x \leq y \Rightarrow z + x \leq z + y$ for all $x, y, z \in R$,

(ii) $x \leq y \Rightarrow z \cdot x \leq z \cdot y$ for all $x, y, z \in R$ where $z \geq 0$.

For example the ordered field $\mathbb{R}$ of reals.

So it looks like, we can offically define a mathematical structure as a quadruple $\underline{S} := (S, \mathcal{C}, \mathcal{F}, \mathcal{R})$, where S is a set, $\mathcal{C}$ is a set of constants from S (possibly empty), $\mathcal{F}$ is a (possibly empty) set of operations on S (possibly of different arity = [illegible]), and $\mathcal{R}$ is a set of relations on S (again possibly empty).
This is an awkward definition because it doesn't provide names for the constants, operations, and relations, hence makes it inconvenient to write axioms that we want them to satisfy. Thus, we first introduce a naming system/format, and then define structures in a given format.

Def. A *signature* or *language* is a quadruple $\sigma := (\mathcal{C}, \mathcal{F}, \mathcal{R}, a)$, where $\mathcal{C}, \mathcal{F}, \mathcal{R}$ are sets of *symbols* (i.e. meaningless characters, e.g. ♡) or *names*, and $a : \mathcal{F} \cup \mathcal{R} \to \mathbb{N}^+ := \{1, 2, 3, \dots\}$ called *arity* (= [illegible]). We call $\mathcal{C}, \mathcal{F}, \mathcal{R}$ the sets of *constant symbols*, *function symbols*, and *relation symbols*. These might be empty.