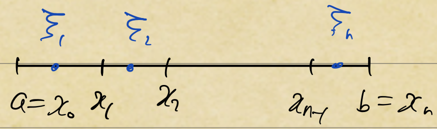
Որոշյալ ինտեգրալ

$f:[a,b]\to\mathbb{R}, \quad (P,\xi)$



$$\sigma(f,P,\xi)=\sum_{k=1}^{n} f(\xi_k)\Delta x_k, \qquad \Delta_k=[x_{k-1},x_k], \quad \Delta x_k = x_k - x_{k-1}$$

$$f\in R[a,b] \overset{def}{\Longleftrightarrow} \exists \lim_{\substack{\lambda(P)\to 0 \\ \parallel \\ \max\limits_{1\le k\le n}(x_k-x_{k-1})}} \sigma(f,P,\xi) =: \int_a^b f(x)\,dx$$

Թեորեմ Դիցուք տրված է $f:[a,b]\to\mathbb{R}$ սահմանափակ ֆունկցիան: Այդժամ

$$f\in R[a,b] \Longleftrightarrow \forall\varepsilon>0\ \exists\delta>0\ \forall P\quad \lambda(P)<\delta \Rightarrow \sum_{k=1}^{n}\omega(f,\Delta_k)\Delta x_k<\varepsilon$$

Թեորեմ (Լեբեգի հայտանիշ)
Դիցուք տրված է $f:[a,b]\to\mathbb{R}$: Այդժամ

$f\in R[a,b] \Longleftrightarrow f$ սահմանափակ է և համարյա ամենուրեք անընդհատ:

Թեորեմ ա) $f,g\in R[a,b] \Rightarrow \forall c\in\mathbb{R}\quad f+cg\in R[a,b]$ և

$$\int_a^b (f+cg)\,dx = \int_a^b f\,dx + c\int_a^b g\,dx$$

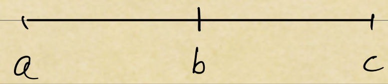
բ) $f,g\in R[a,b],\ f\le g \Rightarrow \int_a^b f\,dx \le \int_a^b g\,dx$

գ) $f,g\in R[a,b] \Rightarrow fg\in R[a,b]$

դ) $f\in R[a,b] \Rightarrow \forall [c,d]\subset[a,b]\quad f\in R[c,d]$

ե) $f:[a,c]\to\mathbb{R},\ b\in(a,c),\ f|_{[a,b]}\in R[a,b],\ f|_{[b,c]}\in R[b,c]$: Այդժամ

$$f \in R[a,c] \quad \& \quad \int_a^c f\,dx = \int_a^b f\,dx + \int_b^c f\,dx$$



Ապացույց ա) $f+g \in R[a,b]$, $cg \in R[a,b]$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall P \quad \lambda(P) < \delta \Rightarrow \sum_{k=1}^{n} \omega(f+g, \Delta_k)\Delta x_k < \varepsilon$$

$$\omega(f+g, \Delta) \leq \omega(f, \Delta) + \omega(g, \Delta)$$

$$\omega(f+g, \Delta) = \sup_{x, y \in \Delta} |f(x) + g(x) - f(y) - g(y)|$$

$$\leq \sup_{x, y \in \Delta} \left(|f(x) - f(y)| + |g(x) - g(y)|\right)$$

$$\leq \sup_{x, y \in \Delta} |f(x) - f(y)| + \sup_{x, y \in \Delta} |g(x) - g(y)| = \omega(f, \Delta) + \omega(g, \Delta)$$

$$\int_a^b (f+g)\,dx = \lim_{\lambda(P) \to 0} \sigma(f+g, P, \xi) = \lim_{\lambda(P) \to 0} \left(\sigma(f, P, \xi) + \sigma(g, P, \xi)\right)$$

$$= \int_a^b f\,dx + \int_a^b g\,dx$$

բ) • $f \geq 0$, $f \in R[a,b] \Rightarrow \int_a^b f\,dx \geq 0$

$f, g \in R[a,b]$, $f \leq g \to h = g - f \geq 0 \Rightarrow \int_a^b h\,dx \geq 0 \Rightarrow \int_a^b g\,dx \geq \int_a^b f\,dx$

• $$\int_a^b f\,dx = \lim_{\lambda(P) \to 0} \underbrace{\sum_{k=1}^{n} f(\xi_k)\Delta x_k}_{\geq 0} \geq 0$$

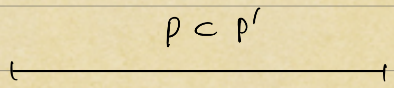
գ) • $f \in R[a,b] \Rightarrow f^2 \in R[a,b]$

$$\sum_{k=1}^{n} \omega(f^2, \Delta_k)\Delta x_k \leq C \overbrace{\sum_{k=1}^{n} \omega(f, \Delta_k)\Delta x_k}^{\leq \varepsilon/C \text{ երբ } \lambda(P) < \delta} \leq \varepsilon$$

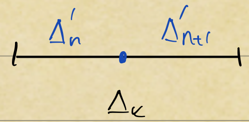
$$|f^2(x) - f^2(y)| \leq \left(2 \sup_{[a,b]} |f|\right) |f(x) - f(y)|$$

$$fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2 \in R[a,b]$$

դ, ե)

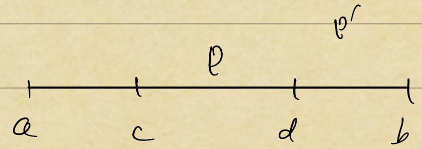


$$\sum_{\Delta_k \in P} \omega(f, \Delta_k)\Delta x_k \geq \sum_{\Delta'_k \in P'} \omega(f, \Delta'_k)\Delta x'_k$$



$$\omega(f, \Delta_k) \leq \omega(f, \Delta'_n) + \omega(f, \Delta'_{n+1})$$

$$\omega(f, \Delta_k) \geq \omega(f, \Delta'_n), \ \omega(f, \Delta'_{n+1})$$



Հետևանք

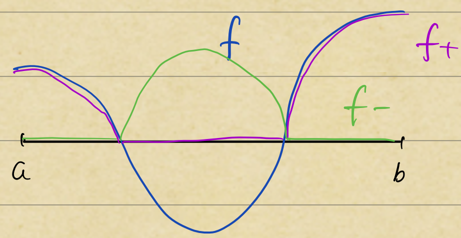
ա) $f \in R[a,b] \Rightarrow |f| \in R[a,b], \ \left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$

բ) $f \in R[a,b], \ F(x) = \int_a^x f(t)\,dt \Rightarrow \forall x,y \in [a,b] \ |F(x) - F(y)| \leq C|x-y|$

որտեղ $C = \sup_{[a,b]} |f|$ ։

Ապացույց

ա)



$$f = f_+ - f_- \qquad\qquad f_+ = \frac{1}{2}(|f| + f)$$

$$|f| = f_+ + f_- \iff f_- = \frac{1}{2}(|f| - f)$$

$$f_+ = \max(f, 0), \quad f_- = \max(-f, 0)$$

$$\left|\int_a^b f\,dx\right| = \left|\int_a^b f_+\,dx - \int_a^b f_-\,dx\right| \leq \int_a^b f_+\,dx + \int_a^b f_-\,dx = \int_a^b \underbrace{(f_+ + f_-)}_{|f|}\,dx$$

բ) $\sup_{[a,b]} |f| = C \Rightarrow \left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx \leq \int_a^b C\,dx$

$$= C(b-a)$$

$$x < y \Rightarrow \left|\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right|$$

$$= \left|\int_x^y f(t)\,dt\right| \leq \left(\sup_{[x,y]} |f|\right)(y-x) \leq C|y-x|$$

Որոշյալ ինտեգրալի կիրառությունները

անհավասարության գործակիցների համար $\Rightarrow$ Հաշվել Եթե 5 ինտեգրալի համար

Օրինակ 1) $a_k, b_k, \quad k=1,\dots,n, \qquad b_k \geq 0$

$$\underbrace{\left(\min_{1\le k\le n} a_k\right)}_{m} \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{n} a_k b_k \leq \underbrace{\left(\max_{k=1,\dots,n} a_k\right)}_{M} \sum_{k=1}^{n} b_k$$

$$\forall k \in [\![1,n]\!] \qquad m \leq a_k \leq M \quad \Rightarrow \quad m b_k \leq a_k b_k \leq M b_k$$

2) $$m \leq f \leq M \quad \Rightarrow \quad m(b-a) \leq \int_a^b f\,dx \leq M(b-a)$$

$$\Rightarrow \boxed{C \in [m,M] \qquad \int_a^b f\,dx = C(b-a)}$$

$$f \in C([a,b]) \quad \Rightarrow \quad M = \max f, \quad m = \min f; \quad \forall C \in [m,M]$$
$$\exists \xi \in [a,b] \quad f(\xi) = C$$

$$C \in [f(a), f(b)] \Rightarrow \exists \xi \in [a,b] \quad f(\xi) = C$$

$$\exists \xi \in [a,b] \qquad \int_a^b f\,dx = f(\xi)(b-a)$$

Թեորեմ Դիցուք $f, g \in R[a,b]$, ընդ որում $g \geq 0$ կամ $g \leq 0$: Ապա

$$\exists C \in [\underset{\underset{[a,b]}{\inf f}}{m_f}, \underset{\underset{[a,b]}{\sup f}}{M_f}] \qquad \int_a^b fg\,dx = C \int_a^b g\,dx$$

Եթե $f \in C([a,b])$, ապա $\exists \xi \in [a,b]$ $\quad \int_a^b fg\,dx = f(\xi) \int_a^b g\,dx$:

Օրինակ $$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} \Rightarrow \int_0^1 f\,dx = 0$$

$\exists C \in (0,1) \quad \int_0^1 f\,dx = C$ սխալ է

Ապացույց $$\sigma(fg, P, \xi) = \sum_{k=1}^{n} f(\xi_k) g(\xi_k) \Delta x_k \leq M_f\, \sigma(g, P, \xi)$$

$$\sigma(fg, P, \xi) \geq m_f\, \sigma(g, P, \xi)$$

Անցնենք սահմանին, երբ $\lambda(P) \to 0$.

$$m_f \int_a^b g\,dx \le \int_a^b fg\,dx \le M_f \int_a^b g\,dx$$

$$\int g\,dx > 0 \Rightarrow m_f \le \underbrace{\frac{\int fg\,dx}{\int g\,dx}}_{= c \in [m_f, M_f]} \le M_f$$

Άσκ α) $g \in R[a,b]$, $g \ge 0$; $\exists x_0 \in [a,b]$ $g(x_0) > 0$ και g η συνεχής στο x_0 [illegible]: Τότε

$$\int_a^b g\,dx > 0$$

β) $g \in R[a,b]$, $g \ge 0$, $\int_a^b g\,dx = 0 \Rightarrow \exists A \subset [a,b]$ $\mathcal{L}(A) = 0$ και $g(x) = 0 \quad \forall x \notin A$

γ) $g \in R[a,b]$, $g = 0$ [illegible] [illegible] $\Rightarrow \int_a^b g\,dx = 0$:

$$g_\varepsilon = g + \varepsilon \ge \varepsilon, \quad \int g_\varepsilon\,dx > 0$$

$$\int_a^b f g_\varepsilon\,dx = c_\varepsilon \int_a^b g_\varepsilon\,dx, \quad c_\varepsilon \in [m_f, M_f]$$

$$\int_a^b g_\varepsilon\,dx = \int_a^b g\,dx + \varepsilon(b-a) \xrightarrow[\varepsilon \to 0]{} \int_a^b g\,dx$$

$$\int_a^b f g_\varepsilon\,dx = \int_a^b fg\,dx + \varepsilon \int_a^b f\,dx \to \int_a^b fg\,dx$$

$$\exists\, \varepsilon_k \searrow 0 \quad c_{\varepsilon_k} \to c \in [m_f, M_f]$$